

GEOMETRIC SCHUR DUALITY OF TWO PARAMETER QUANTUM GROUP OF TYPE A

HAITAO MA, ZONGZHU LIN, AND ZHU-JUN ZHENG

ABSTRACT. In this paper, we give an geometric description of the Schur-Weyl duality for two-parameter quantum algebras $U_{v,t}(gl_n)$, where $U_{v,t}(gl_n)$ is the deformation of $U_v(I, \cdot)$, the classic Schur-Weyl duality $(U_{r,s}(gl_n), V^{\otimes d}, H_d(r, s))$ can be seen as a corollary of the Schur-Weyl duality $(U_{v,t}(gl_n), V^{\otimes d}, H_d(v, t))$ by using the galois descend approach. we also establish the Schur-Weyl duality between the algebras $U_{v,t}(\widetilde{gl_N})^m$, $U_{v,t}(\widetilde{gl_N})^m$ and Heck algebra $H_k(v, t)$.

1. INTRODUCTION

Schur-Weyl duality is a classical method to construct irreducible modules of simple Lie groups out of the fundamental representations [W46], The quantum version for the quantum enveloping algebra $U_q(sl_n)$ and the Hecke algebra $H_q(S_m)$ has been one of the pioneering examples [13] in the fervent development of quantum groups. Two-parameter general linear and special linear quantum groups [21, 8, 4] are certain generalization of the one-parameter Drinfeld-Jimbo quantum groups [7, 12]. The two-parameter quantum groups also had their origin in the quantum inverse scattering method [20] as well as other approaches [14, 6]. So far, lots of mathematicians had studied the quantum groups and two parameter quantum group. For example, geometric Schur-Jimbo duality of type A was studied by Beilinson, Lusztig and Mcpherson [BLM90]. And the Schur-like duality of type B/C and D were discovered by Bao-Wang [BKLW14] and Fan-Li [FL14].

Especially, Fan and Li had found another version of two parameter quantum group by the way of perverse sheaves [FL13]. But the question how the two parameter quantum group $U_{v,t}(gl_n)$ can be seen as the deformation of $U_v(gl_n)$ didn't solve in their work. So it is necessary for us to give the new graded structure on $U_v(gl_n)$ such that $U_{v,t}(gl_n)$ can be seen as the deformation of $U_v(gl_n)$.

Fan and Li found two new quantum group \mathbf{U} and \mathbf{U}^m , and gave the Schur-Weyl duality between them and the Iwahori-hecke algebra of type D_d [FL14]. In our following paper, similar to the Fan and Li's work, we will give two new two parameter quantum group $\mathbf{U}_{v,t}$ and $\mathbf{U}_{v,t}^m$. We can also give the Schur-Weyl duality between them and the two parameter Iwahori-hecke algebra of type D_d through the geometric way. In order to give the comultiplication of the two new two parameter quantum group $\mathbf{U}_{v,t}$, $\mathbf{U}_{v,t}^m$ and use the comultiplication structure to give the Schur-Weyl duality algebraically. That is,

$$\Delta : \mathbf{U} \rightarrow U_{v,t}(\widetilde{gl_N})^m \otimes \mathbf{U},$$

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$$\Delta : \mathbf{U}^m \rightarrow U_{v,t}(\widehat{gl_N})^m \otimes \mathbf{U}^m.$$

So it is reasonable for us to give structure of the new quantum group $U_{v,t}(\widehat{gl_N})^m$, $U_{v,t}(\widehat{gl_N})^m$ and the Shur-Weyl duality between them and $H_{v,t}(d)$.

In this work, at first, we give a new version of two parameter quantum group $U_{v,t}(gl_n)$, which is the deformation of $U_v(gl_n)$ similar to the approach appear in [FL13]. Second, we would like to give the geometric realization of three quantum groups $U_{v,t}(gl_n)$, $U_{v,t}(\widehat{gl_N})^m$, $U_{v,t}(\widehat{gl_N})^m$. At the same time, we also give the Shur-Weyl duality between algebras $U_{v,t}(gl_n)$, $U_{v,t}(\widehat{gl_N})^m$, $U_{v,t}(\widehat{gl_N})^m$ and the Hecke algebra $H_{v,t}(d)$. Since the classical two parameter quantum group $U_{r,s}(gl_n)$ is the subalgebra of the new version $U_{v,t}(gl_n)$. we would like to use the Galois descend approach to understand the two different versions of two parameter quantum groups. The classical Shur-Weyl duality $(U_{r,s}(gl_n), V^{\otimes d}, H_d(r, s))$ can be seen as a corollary of the Shur-Weyl duality $(U_{v,t}(gl_n), V^{\otimes d}, H_d(v, t))$ by using the galois descend theory. That is, there exist a Galois group G such that $(U_{v,t}(gl_n)^G, (V^{\otimes d})^G, H_d(v, t)^G)$ is also Shur-Weyl duality, and $U_{v,t}(gl_n)^G \cong (U_{r,s}(gl_n), H_d(v, t)^G \cong H_d(r, s)$.

2. DEFORMATION

2.1. **The algebra $U_{v,t}(gl_n)$.** let $\Omega = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}_{n \times n}$ is associated to the

cartan matrix of type A_n . Let $I = \{1, 2, \dots, n\}$. To Ω , we associate the following three bilinear forms on \mathbb{Z}^I .

$$\begin{aligned} (1) \quad \langle i, j \rangle &= \Omega_{ij}, & \forall i, j \in I. \\ (2) \quad [i, j] &= 2\delta_{ij}\Omega_{ii} - \Omega_{ij}, & \forall i, j \in I. \\ (3) \quad i \cdot j &= \langle i, j \rangle + \langle j, i \rangle, & \forall i, j \in I. \end{aligned}$$

Definition 2.1.1. *The two-parameter quantum algebra $U_{v,t}(gl_n)$ associated to A_{n-1} is an associative $\mathbb{Q}(v, t)$ -algebra with 1 generated by symbols $E_i, F_i, \forall i \in I$ $A_j^{\pm 1}, B_j^{\pm 1}, \forall i \in I' =$*

$I \cup \{n\}$ and subject to the following relations.

$$\begin{aligned}
(R1) \quad & A_i^{\pm 1} A_j^{\pm 1} = A_j^{\pm 1} A_i^{\pm 1}, \quad B_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} B_i^{\pm 1}, \\
& A_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} A_i^{\pm 1}, \quad A_i^{\pm 1} A_i^{\mp 1} = 1 = B_i^{\pm 1} B_i^{\mp 1}. \\
(R2) \quad & A_i E_j A_i^{-1} = v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \quad B_i E_j B_i^{-1} = v^{-\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \\
& A_i F_j A_i^{-1} = v^{-\langle i, j \rangle} t^{-\langle j, i \rangle} F_j, \quad B_i F_j B_i^{-1} = v^{\langle i, j \rangle} t^{-\langle j, i \rangle} F_j. \\
(R3) \quad & E_i F_j - F_j E_i = \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{v - v^{-1}}. \\
(R4) \quad & \sum_{p+p'=1-2\frac{i \cdot j}{i \cdot i}} (-1)^p t^{-p(p'-2\frac{\langle i, j \rangle}{i \cdot i} + 2\frac{\langle j, i \rangle}{i \cdot i})} E_i^{(p')} E_j E_i^{(p)} = 0, \quad \text{if } i \neq j, \\
& \sum_{p+p'=1-2\frac{i \cdot j}{i \cdot i}} (-1)^p t^{-p(p'-2\frac{\langle i, j \rangle}{i \cdot i} + 2\frac{\langle j, i \rangle}{i \cdot i})} F_i^{(p)} F_j F_i^{(p')} = 0, \quad \text{if } i \neq j,
\end{aligned}$$

where $E_i^{(p)} = \frac{E_i^p}{[p]_{v_i, t_i}!}$, $\langle j, n \rangle = 0$, $\langle n, j \rangle = \begin{cases} -1 & \text{if } j = n-1; \\ 0 & \text{else.} \end{cases}$, $j \in I$.

The algebra $U_{v,t}(gl_n)$ has a Hopf algebra structure with the comultiplication Δ , the counit ε and the antipode S given as follows.

$$\begin{aligned}
\Delta(A_i^{\pm 1}) &= A_i^{\pm 1} \otimes A_i^{\pm 1}, & \Delta(B_i^{\pm 1}) &= B_i^{\pm 1} \otimes B_i^{\pm 1}, \\
\Delta(E_i) &= E_i \otimes A_i B_{i+1} + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + B_i A_{i+1} \otimes F_i, \\
\varepsilon(A_i^{\pm 1}) &= \varepsilon(B_i^{\pm 1}) = 1, & \varepsilon(E_i) &= \varepsilon(F_i) = 0, & S(A_i^{\pm 1}) &= A_i^{\mp 1}, \\
S(B_i^{\pm 1}) &= B_i^{\mp 1}, & S(E_i) &= -E_i B_i A_{i+1}, & S(F_i) &= -A_i B_{i+1} F_i.
\end{aligned}$$

The algebra $U_{v,t}(gl_n)$ admits a $\mathbb{Z}^{I'} \times \mathbb{Z}^{I'}$ -grading by defining the degrees of generators as follows.

$$\deg(E_i) = (i, 0), \quad \deg(F_i) = (0, i),$$

$$\deg(A_j) = \deg(B_j) = \begin{cases} (\sum_{k=j}^n (-1)^k k, \sum_{k=j}^n (-1)^k k) & \text{if } j \text{ is even,} \\ (\sum_{k=j}^n (-1)^{k+1} k, \sum_{k=j}^n (-1)^{k+1} k) & \text{if } j \text{ is odd.} \end{cases}$$

We can define a bilinear form on $\mathbb{Z}^{I'} \times \mathbb{Z}^{I'}$ by

$$[\gamma, \eta]' = [\gamma_2, \eta_2] - [\gamma_1, \eta_1]$$

for any $\gamma = (\gamma_1, \gamma_2), \eta = (\eta_1, \eta_2) \in \mathbb{Z}^{I'} \times \mathbb{Z}^{I'}$. Then on $U_{v,t}(gl_n)$, we can define a new multiplication " $*$ " by

$$(4) \quad x * y = t^{-[|x|, |y|]'} xy,$$

for any homogenous elements $x, y \in U_{v,t}(gl_n)$. Since $[,]'$ is a bilinear form, $(U_{v,t}(gl_n), *)$ is an associative algebra over $\mathbb{Q}(v, t)$. We define a multiplication, denoted by " $*$ ", on

$U_{v,t}(gl_n) \otimes U_{v,t}(gl_n)$ by

$$(5) \quad (x \otimes y) * (x' \otimes y') = x * x' \otimes y * y'.$$

This gives a new algebra structure on $U_{v,t}(gl_n) \otimes U_{v,t}(gl_n)$. $(U_{v,t}(gl_n), *)$ has a Hopf algebra structure with the comultiplication Δ^* , the counit ε^* and the antipode S^* . The image of generators E_i, F_i, A_i and B_i^{-1} under the map Δ^* (resp. ε^* and S^*) are the same as the ones under the map Δ (resp. ε and S) defined above.

Lemma 2.1.2. *Under the new multiplication “*”, the defining relations of $U_{v,t}(gl_n)$ can be rewritten as follows.*

$$\begin{aligned} (R^*1) \quad & A_i^{\pm 1} * A_j^{\pm 1} = A_j^{\pm 1} * A_i^{\pm 1}, \quad B_i^{\pm 1} * B_j^{\pm 1} = B_j^{\pm 1} * B_i^{\pm 1}, \\ & A_i^{\pm 1} * B_j^{\pm 1} = B_j^{\pm 1} * A_i^{\pm 1}, \quad A_i^{\pm 1} * A_i^{\mp 1} = 1 = B_i^{\pm 1} * B_i^{\mp 1}. \\ (R^*2) \quad & A_i * E_j * A_i^{-1} = v^{\langle i, j \rangle} E_j, \quad B_i * E_j * B_i^{-1} = v^{-\langle i, j \rangle} E_j, \\ & A_i * F_j * A_i^{-1} = v^{-\langle i, j \rangle} F_j, \quad B_i * F_j * B_i^{-1} = v^{\langle i, j \rangle} F_j. \\ (R^*3) \quad & E_i * F_j - F_j * E_i = \delta_{ij} \frac{A_i * B_{i+1} - B_i * A_{i+1}}{v - v^{-1}}, \quad \forall i, j \in I. \\ (R^*4) \quad & \sum_{p+p'=1-a_{ij}} (-1)^p \begin{bmatrix} 1 - a_{ij} \\ p \end{bmatrix}_v E_i^{*p} * E_j * E_i^{*p'} = 0, \quad \text{if } i \neq j, \\ & \sum_{p+p'=1-a_{ij}} (-1)^p \begin{bmatrix} 1 - a_{ij} \\ p \end{bmatrix}_v F_i^{*p} * F_j * F_i^{*p'} = 0 \quad \text{if } i \neq j, \end{aligned}$$

where $a_{ij} = 2\frac{i \cdot j}{i \cdot i}$ and $E_i^{*p} = E_i * E_i * \cdots * E_i$ for p copies. We notice that these relations are the specialization of (R1)-(R4) at $t = 1$.

Proof. The relation R3, R4 agrees with the one in [FL13, 4. 2], whose proof is also the same as the one for type-**A** case. Next we show R^*2 .

$$A_i * E_j = t^{-[|A_i|, |E_j|]'} A_i E_j = t^{-[|A_i|, |E_j|]'} v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j A_i = t^{[|E_j|, |A_i|]' - [A_i|, |E_j|]'} v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j * A_i$$

and

$$[|E_j|, |A_i|]' - [A_i|, |E_j|]' = (\langle j, i \rangle - \langle i, j \rangle) - (\langle j, i+1 \rangle - \langle i+1, j \rangle) + \cdots + (-1)^{n-i} (\langle j, n \rangle - \langle n, j \rangle).$$

$$[|E_j|, |A_i|]' - [A_i|, |E_j|]' = \begin{cases} = \langle i+1, j \rangle = 1 = -\langle i, j \rangle & \text{if } i = j, \\ = (\langle j, i \rangle - \langle i, j \rangle) - (\langle j, i+1 \rangle - \langle i+1, j \rangle) = 1 = -\langle i, j \rangle & \text{if } i = j+1, \\ = 0 = -\langle i, j \rangle & \text{if } i - j > 1, \\ = \langle j, i \rangle - \langle i+1, j \rangle = 0 = -\langle i, j \rangle & \text{if } j = i+1, \\ = -\langle j-1, j \rangle + \langle j, j-1 \rangle = 0 = -\langle i, j \rangle & \text{if } j - i > 1. \end{cases}$$

Therefore,

$$A_i * E_j = v^{\langle i, j \rangle} E_j * A_i.$$

All other identity in R2 can be shown similarly. \square

The one-parameter quantum algebra $U_v(I, \cdot)$ associated to (I, \cdot) is defined as the associative $\mathbb{Q}(v)$ -algebra with 1 generated by symbols $E_i, F_i, A_i^{\pm 1}, B_i^{\pm 1}, \forall i \in I$ and subject to relations (R*1)-(R*4). $U_v(I, \cdot)$ has a Hopf algebra structure with the comultiplication Δ_1 , the counit ε_1 and the antipode S_1 . The image of generators E_i, F_i, A_a, B_a under the map Δ_1 (resp. ε_1 and S_1) are the same as the ones under the map Δ (resp. ε and S) defined above.

Let $U_{v,t}(I, \cdot) := U_v(I, \cdot) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v, t)$. The Hopf algebra structure on $U_v(I, \cdot)$ can be naturally extended to $U_{v,t}(I, \cdot)$. From the above analysis, we have the following theorem.

Theorem 2.1.3. *If (I, \cdot) is the Cartan datum associated to Ω_n , then there is a Hopf-algebra isomorphism*

$$(U_{v,t}(gl_n), *, \Delta^*, \varepsilon^*, S^*) \simeq (U_{v,t}(I, \cdot), \cdot, \Delta_1, \varepsilon_1, S_1),$$

sending the generators in $U_{v,t}$ to the respective generators in $U_{v,t}(I, \cdot)$.

3. A GEOMETRIC SETTING

3.1. Preliminary. Let \mathbb{F}_q be a finite field of q elements and of odd characteristic. d is a fixed positive integer,

n is a positive integer, We fix a vector space \mathbb{F}_q^d . Consider the following sets.

- The set \mathcal{X} of n -step flags $V = (V_i)_{0 \leq i \leq n}$ in \mathbb{F}_q^d such that $V_0 = 0, V_i \subseteq V_{i+1}$.
- The set \mathcal{Y} of complete flags $F = (F_i)_{0 \leq i \leq d}$ in \mathbb{F}_q^d such that $F_i \subset F_{i+1}, |F_i| = i$.

where we write $|F_i|$ for the dimension of F_i .

Let $G = \text{GL}(V)$. Then G acts naturally on sets \mathcal{X} and \mathcal{Y} . Moreover, G acts transitively on \mathcal{Y} . Let G act diagonally on the product $\mathcal{X} \times \mathcal{X}$ (resp. $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{Y} \times \mathcal{Y}$). Set

$$(6) \quad \mathcal{A} = \mathbb{Z}[v^{\pm 1}, t^{\pm 1}].$$

Let

$$(7) \quad \mathcal{S}_{\mathcal{X}} = \mathcal{A}_G(\mathcal{X} \times \mathcal{X})$$

be the set of all \mathcal{A} -valued G -invariant functions on $\mathcal{X} \times \mathcal{X}$. Clearly, the set $\mathcal{S}_{\mathcal{X}}$ is a free \mathcal{A} -module. Moreover, $\mathcal{S}_{\mathcal{X}}$ admits an associative \mathcal{A} -algebra structure ‘ $*$ ’ under a standard convolution product as discussed in [BKLW14, 2. 3]. In particular, when v is specialized to \sqrt{q} , we have

$$(8) \quad f * g(V, V') = \sum_{V'' \in \mathcal{X}} f(V, V'')g(V'', V'), \quad \forall V, V' \in \mathcal{X}.$$

Similarly, we define the free \mathcal{A} -modules

$$(9) \quad \mathcal{V} = \mathcal{A}_G(\mathcal{X} \times \mathcal{Y}) \quad \text{and} \quad \mathcal{H}_{\mathcal{Y}} = \mathcal{A}_G(\mathcal{Y} \times \mathcal{Y}).$$

A similar convolution product gives an associative algebra structure on $\mathcal{H}_{\mathcal{Y}}$ and a left $\mathcal{S}_{\mathcal{X}}$ -action and a right $\mathcal{H}_{\mathcal{Y}}$ -action on \mathcal{V} . Moreover, these two actions commute and hence we have the following \mathcal{A} -algebra homomorphisms.

$$\mathcal{S}_{\mathcal{X}} \rightarrow \text{End}_{\mathcal{H}_{\mathcal{Y}}}(\mathcal{V}) \quad \text{and} \quad \mathcal{H}_{\mathcal{Y}} \rightarrow \text{End}_{\mathcal{S}_{\mathcal{X}}}(\mathcal{V}).$$

Similar to [P09, Theorem 2. 1], we have the following double centralizer property.

Lemma 3.1.1. *$\text{End}_{\mathcal{H}_{\mathcal{Y}}}(\mathcal{V}) \simeq \mathcal{S}_{\mathcal{X}}$ and $\text{End}_{\mathcal{S}_{\mathcal{X}}}(\mathcal{V}) \simeq \mathcal{H}_{\mathcal{Y}}$, if $n \geq d$.*

We note that the result in [P09, Theorem 2. 1] is obtained over the field \mathbb{C} of complex numbers, but the proof can be adapted to our setting over the ring \mathcal{A} .

We shall give a description of the G -orbits on $\mathcal{X} \times \mathcal{X}$, $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{Y} \times \mathcal{Y}$. We start by introducing the following notations associated to a matrix $M = (m_{ij})_{1 \leq i, j \leq c}$.

$$(10) \quad \begin{aligned} \text{ro}(M) &= \left(\sum_{j=1}^c m_{ij} \right)_{1 \leq i \leq c}, \\ \text{co}(M) &= \left(\sum_{i=1}^c m_{ij} \right)_{1 \leq j \leq c}. \end{aligned}$$

We also write $\text{ro}(M)_i$ and $\text{co}(M)_j$ for the i -th and j -th component of the row vectors of $\text{ro}(M)$ and $\text{co}(M)$, respectively.

For any pair (V, V') of flags in \mathcal{X} , we can assign an n by n matrix whose (i, j) -entry equal to $\dim \frac{V_{i-1} + V_i \cap V'_j}{V_{i-1} + V_i \cap V'_{j-1}}$.

$$(11) \quad G \backslash \mathcal{X} \times \mathcal{X} \simeq \Theta_d,$$

where Θ_d is the set of all matrices Θ_d in $\text{Mat}_{n \times n}(\mathbb{N})$ such that $\sum_{i,j} (\Theta_d)_{i,j} = d$

A similar assignment yields two bijection

$$(12) \quad G \backslash \mathcal{X} \times \mathcal{Y} \simeq \Pi,$$

$$(13) \quad G \backslash \mathcal{Y} \times \mathcal{Y} \simeq \Sigma,$$

where the set Π consists of all matrices $B = (b_{ij})$ in $\text{Mat}_{n \times d}(\mathbb{N})$ subject to

$$\text{co}(B)_j = 1, \quad \forall j \in [1, d].$$

and Σ is the set of all matrices $\sigma \equiv (\sigma_{ij})$ in $\text{Mat}_{d \times d}(\mathbb{N})$ such that

$$\text{ro}(\sigma)_i = 1, \quad \text{ro}(\sigma)_j = 1.$$

Moreover, we have

$$(14) \quad \#\Sigma = d! \quad \text{and} \quad \#\Pi = n^d.$$

4. CALCULUS OF THE ALGEBRA \mathcal{S} AND $\mathcal{H}_{\mathcal{Y}}$

Recall from the previous section that $\mathcal{S}_{\mathcal{X}}$ is the convolution algebra on $\mathcal{X} \times \mathcal{X}$ defined in (7). For simplicity, we shall denote \mathcal{S} instead of $\mathcal{S}_{\mathcal{X}}$. In this section, we determine the generators for \mathcal{S} and the associated multiplication formula. We also will $\mathcal{H}_{\mathcal{Y}}$ action on \mathcal{V} .

4.1. **Defining relations of \mathcal{S} .** For any $i \in [1, n-1]$, $a \in [1, n]$, set

$$(15) \quad \begin{aligned} E_i(V, V') &= \begin{cases} v^{-|V'_i/V'_{i-1}|} t^{-|V_i/V_{i-1}|}, & \text{if } V_i \supset^1 V'_i, V_j = V_{j'}, \forall j \in [1, n] \setminus \{i\}; \\ 0, & \text{otherwise.} \end{cases} \\ F_i(V, V') &= \begin{cases} v^{-|V'_{i+1}/V'_i|} t^{|V'_{i+1}/V'_i|}, & \text{if } V_i \subset^1 V'_i, V_j = V_{j'}, \forall j \in [1, n] \setminus \{i\}; \\ 0, & \text{otherwise.} \end{cases} \\ A_a^{\pm 1}(V, V') &= \begin{cases} v^{\pm |V'_a/V'_{a-1}|} t^{\pm |V'_a/V'_{a-1}|}, & \text{if } V = V'; \\ 0, & \text{otherwise.} \end{cases} \\ B_a^{\pm 1}(V, V') &= \begin{cases} v^{\mp |V'_a/V'_{a-1}|} t^{\pm |V'_a/V'_{a-1}|}, & \text{if } V = V'; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that these functions are elements in \mathcal{S} .

Proposition 4.1.1. *The functions $E_i, F_i, A_a^{\pm 1}$ and $B_a^{\pm 1}$ in \mathcal{S} , for any $i \in [1, n-1]$, $a \in [1, n]$, satisfy the following relations.*

$$\begin{aligned} (R1) \quad & A_i^{\pm 1} A_j^{\pm 1} = A_j^{\pm 1} A_i^{\pm 1}, \quad B_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} B_i^{\pm 1}, \\ & A_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} A_i^{\pm 1}, \quad A_i^{\pm 1} A_i^{\mp 1} = 1 = B_i^{\pm 1} B_i^{\mp 1}. \\ (R2) \quad & A_i E_j A_i^{-1} = v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \quad B_i E_j B_i^{-1} = v^{-\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \\ & A_i F_j A_i^{-1} = v^{-\langle i, j \rangle} t^{-\langle j, i \rangle} F_j, \quad B_i F_j B_i^{-1} = v^{\langle i, j \rangle} t^{-\langle j, i \rangle} F_j. \\ (R3) \quad & E_i F_j - F_j E_i = \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{v - v^{-1}}. \\ (R4) \quad & \sum_{p+p'=1-2\frac{i,j}{i,i}} (-1)^p t^{-p(p'-2\frac{\langle i,j \rangle}{i,i}+2\frac{\langle j,i \rangle}{i,i})} E_i^{(p')} E_j E_i^{(p)} = 0, \quad \text{if } i \neq j, \\ & \sum_{p+p'=1-2\frac{i,j}{i,i}} (-1)^p t^{-p(p'-2\frac{\langle i,j \rangle}{i,i}+2\frac{\langle j,i \rangle}{i,i})} F_i^{(p)} F_j F_i^{(p')} = 0, \quad \text{if } i \neq j, \\ (R5) \quad & \prod_{i=1}^n A_i = v^d t^d, \quad \prod_{i=1}^n B_i = v^{-d} t^d, \\ (R6) \quad & \prod_{l=0}^d (A_j - v^l t^l) = 0, \quad \prod_{l=0}^d (B_j - v^{-l} t^l) = 0 \forall j \in [1, n]. \\ (R7) \quad & E_i^{d+1} = 0, F_i^{d+1} = 0. \end{aligned}$$

Proof. The proofs of the identities of R1, R7 are straightforward. Let $\lambda'_i = |V'_i/V'_{i-1}|$. We show the first identity in R2. we have

$$(A_i E_j)(V, V') = \begin{cases} v^{\lambda'_i - \lambda'_j - 1} t^{\lambda'_i + \lambda'_j} & \text{if } V_j \supset^1 V'_j \text{ and } i = j + 1, \\ v^{\lambda'_i - \lambda'_j + 1} t^{\lambda'_i + \lambda'_j + 2} & \text{if } V_j \supset^1 V'_j \text{ and } i = j, \\ v^{\lambda'_i - \lambda'_j} t^{\lambda'_i + \lambda'_j + 1} & \text{if } V_j \supset^1 V'_j \text{ and } i \neq j, j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$(A_i E_j)(V, V') = \begin{cases} v^{\lambda'_i - \lambda'_j} t^{\lambda'_i + \lambda'_j + 1} & \text{if } V_j \supset^1 V'_j, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $A_i E_j A_i^{-1}(V, V') = v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j(V, V')$. All other identities can be shown similarly. we show the identity in R3. By a direct calculation. We have

$$(E_i F_j - F_j E_i)(V, V') = \begin{cases} \frac{v^{\lambda'_i - \lambda'_{i+1}} t^{\lambda'_i + \lambda'_{i+1} - v^{\lambda'_{i+1} - \lambda'_i} t^{\lambda'_i + \lambda'_{i+1}}}}{v - v^{-1}} & \text{if } V = V' \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that the right hand side is equal to $\delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{v - v^{-1}}(V, V')$.

At last, We now show the first identity in R4. By a direct calculation, we have

$$E_i^2 E_{i+1}(V, V') = \begin{cases} (v^2 + 1) v^{-2\lambda'_i - \lambda'_{i+1} - 1} t^{2\lambda'_i + \lambda'_{i+1} + 4}, & \text{if } V_i \supset^2 V'_i \text{ and } V_{i+1} \supset^1 V'_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_i E_{i+1} E_i(V, V') = \begin{cases} (v^2 + 1) v^{-2\lambda'_i - \lambda'_{i+1}} t^{2\lambda'_i + \lambda'_{i+1} + 3}, & \text{if } V_i \supset^2 V'_i \text{ and } V_{i+1} \supset^1 V'_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{i+1} E_i^2(V, V') = \begin{cases} (v^2 + 1) v^{-2\lambda'_i - \lambda'_{i+1} + 1} t^{2\lambda'_i + \lambda'_{i+1} + 2}, & \text{if } V_i \supset^2 V'_i \text{ and } V_{i+1} \supset^1 V'_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The first identity in R4 follows. By the same way, the other three identities can be shown directly.

Let's prove the first identity in R5, we have

$$\prod_{i=1}^n A_i(V, V') = \begin{cases} v^{\lambda'_1 + \dots + \lambda'_n} t^{\lambda'_1 + \dots + \lambda'_n}, & \text{if } V = V', \\ 0, & \text{otherwise.} \end{cases}$$

Since $\lambda'_1 + \dots + \lambda'_n = d$, the first identities follows. The other identities can be shown similarly.

At last, let's prove the first identity in R6, we have

$$\prod_{l=0}^d (A_j - v^l t^l)(V, V') = \begin{cases} (v^{\lambda'_j} t^{\lambda'_j} - 1)(v^{\lambda'_j} t^{\lambda'_j} - vt) \dots (v^{\lambda'_j} t^{\lambda'_j} - v^d t^d), & \text{if } V = V', \\ 0, & \text{otherwise.} \end{cases}$$

Since $0 \leq \lambda'_j \leq d$, the first identities follows. the other identities can be shown similarly. \square

4.2. Multiplication formulas in \mathcal{S} . For any $n \in \mathbb{Z}, k \in \mathbb{N}$, set

$$(n)_v = \frac{v^{2n} - 1}{v^2 - 1}, \quad \text{and} \quad \binom{n}{k}_v = \prod_{i=1}^k \frac{(n+1-i)_v}{(i)_v}.$$

Let E_{ij} is the $n \times n$ matrix whose (i, j) -entry is 1 and all other entries are 0. Let $e_{\mathbf{a}}$ be the characteristic function of the G -orbit corresponding to $\mathbf{a} \in \theta_d$. It is clear that the set $\{e_{\mathbf{a}} | \mathbf{a} \in \theta_d\}$ forms a basis of \mathcal{S} .

We assume that the ground field is an algebraic closure $\overline{\mathbb{F}}_q$ of \mathbb{F}_q when we talk about the dimension of a G -orbit or its stabilizer. Set

$$d(\mathbf{a}) = \dim \mathcal{O}_{\mathbf{a}} \quad \text{and} \quad r(\mathbf{a}) = \dim \mathcal{O}_{\mathbf{b}}, \quad \forall \mathbf{a} \in \theta_d \text{ or } \Pi,$$

where $\mathbf{b} = (b_{ij})$ is the diagonal matrix such that $b_{ii} = \sum_k a_{ik}$. Denote by $C_G(V, V')$ the stabilizer of (V, V') in G .

Lemma 4.2.1. *If $\mathbf{a} \in \Pi$, We have*

$$\begin{aligned} \dim C_G(V, V') &= \sum_{i \geq k, j \geq l} a_{ij} a_{kl}, \quad \text{if } (V, V') \in \mathcal{O}_{\mathbf{a}}, \\ \dim \mathcal{O}_{\mathbf{a}} &= \sum_{i < k \text{ or } j < l} a_{ij} a_{kl}, \\ d(\mathbf{a}) - r(\mathbf{a}) &= \sum_{i \geq k, j < l} a_{ij} a_{kl}. \end{aligned}$$

Proof. The proof is similar with [BLM90], The only difference we consider is that $\mathbf{a} \in \Pi$ should be the $n \times d$ matrix. We can find the subspace Z_{ij} of V such that $V_a = \oplus_{i \leq a, j} Z_{ij}$ for all a , $V'_b = \oplus_{j \leq b, i} Z_{ij}$ for all b . $V = \oplus_{ij} Z_{ij}$. Consider $T \in \text{End}(V)$, T is determined by a family of linear maps $T_{ijkl} : Z_{ij} \rightarrow Z_{kl}$. If $T|_{V_a} = V_a, T|_{V'_b} = V'_b$, one can obtain that if $T_{ijkl} \neq 0$, then $i \geq k, j \geq l$. So we have $\dim C_G(V, V') = \sum_{i \geq k, j \geq l} a_{ij} a_{kl}$, $\dim \mathcal{O}_{\mathbf{a}} = \sum_{i < k \text{ or } j < l} a_{ij} a_{kl}$. Since $r(\mathbf{a}) = \dim(V, V)$, we have $d(\mathbf{a}) - r(\mathbf{a}) = \sum_{i < k \text{ or } j < l} a_{ij} a_{kl} - \sum_{i \geq k, j \geq l} a_{ij} a_{kl} = \sum_{i \geq k, j < l} a_{ij} a_{kl}$. \square

For any $\mathbf{a} \in \theta_d, \Pi$, let

$$\{\mathbf{a}\} = v^{-(d(\mathbf{a})-r(\mathbf{a}))} t^{(d(\mathbf{a})-r(\mathbf{a}))} e_{\mathbf{a}}.$$

We define a bar involution ‘ $-$ ’ on \mathcal{A} by $\bar{v} = v^{-1}$.

Proposition 4.2.2. *Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Theta_d$, $h \in [1, n-1]$ and $r \in \mathbb{N}$.*

(a) If $\text{co}(\mathbf{b}) = \text{ro}(\mathbf{a})$, and $\mathbf{b} - rE_{h,h+1}$ is diagonal, then we have

$$(16) \quad \{\mathbf{b}\} * \{\mathbf{a}\} = \sum_{t: \sum_{u=1}^n t_u = r} v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^n \overline{\binom{a_{hu} + t_u}{t_u}}_v \{\mathbf{a}_t\}, \text{ where}$$

$$\alpha(t) = \sum_{j \geq l} a_{hj} t_l + \sum_{j > l} a_{h+1,j} t_l - \sum_{j < l} t_j t_l$$

$$\beta(t) = \sum_{j \geq l} a_{hj} t_l - \sum_{j > l} a_{h+1,j} t_l + \sum_{j < l} t_j t_l$$

$$\mathbf{a}_t = A + \sum_{u=1}^n t_u (E_{hu} - E_{h+1,u}) \in \theta_d.$$

(b) If $\text{co}(\mathbf{c}) = \text{ro}(\mathbf{a})$ and $\mathbf{c} - rE_{h+1,h}$ is diagonal, then

$$(17) \quad \{\mathbf{c}\} * \{\mathbf{a}\} = \sum_{t: \sum_{u=1}^n t_u = r} v^{\beta'(t)} t^{\alpha'(t)} \prod_{u=1}^n \overline{\binom{a_{h+1,u} + t_u}{t_u}}_v \{\mathbf{a}(h, t)\}, \text{ where}$$

$$\alpha'(t) = \sum_{j \leq l} a_{h+1,j} t_l + \sum_{j < l} a_{hj} t_l - \sum_{j < l} t_j t_l,$$

$$\beta'(t) = \sum_{j \leq l} a_{h+1,j} t_l - \sum_{j < l} a_{hj} t_l + \sum_{j < l} t_j t_l,$$

$$\mathbf{a}(h, t) = A - \sum_{u=1}^n t_u (E_{hu} - E_{h+1,u}) \in \theta_d.$$

Proof. In order to give the proof of (a), We only need to proof the formula $\mathbf{a}(t)$. By the direct computation.

$$d(\mathbf{b}) - r(\mathbf{b}) = \sum_{j,u} a_{h,j} t_u,$$

$$d(\mathbf{a}) - r(\mathbf{a}) = \sum_{i \geq k, j < l} a_{ij} a_{kl},$$

$$d(\mathbf{a}_t) - r(\mathbf{a}_t) = \sum_{i \geq k, j < l} a_{ij} a_{kl} + \sum_{j < u} a_{hj} t_u - \sum_{l > u} a_{h+1,l} t_u + \sum_{u < u'} t_u t_{u'}.$$

Then,

$$\alpha(t) = d(\mathbf{b}) - r(\mathbf{b}) + d(\mathbf{a}) - r(\mathbf{a}) - (d(\mathbf{a}_t) - r(\mathbf{a}_t)) = \sum_{j \geq l} a_{hj} t_l + \sum_{j > l} a_{h+1,j} t_l - \sum_{j < l} t_j t_l.$$

Similarly, we can obtain the proposition of (b). \square

4.3. \mathcal{S} -action on \mathcal{V} . A degenerate version of Proposition 4.2.2 gives us an explicit description of the \mathcal{S} -action on $\mathcal{V} = \mathcal{A}_G(\mathcal{X} \times \mathcal{Y})$ as follows. For any $r_j \in [1, n]$, we denote $\tilde{r}_j = r_j + 1$ and $\hat{r}_j = r_j - 1$.

Corollary 4.3.1. *For any $1 \leq i \leq n-1, 1 \leq a \leq n-1$, we have*

$$\begin{aligned} E_i * \{e_{r_1 \dots r_d}\} &= v^{\sum_{j>p} \delta_{i,r_j} - \delta_{i+1,r_j}} t^{1+\sum_{j>p} \delta_{i,r_j} + \delta_{i+1,r_j}} \{e_{r_1 \dots, \hat{r}_p, \dots r_d}\}, \\ F_i * \{e_{r_1 \dots r_d}\} &= \sum_{1 \leq p \leq d: r_p = i} v^{\sum_{j<p} \delta_{i+1,r_j} - \delta_{i,r_j}} t^{\sum_{j<p} \delta_{i,r_j} + \delta_{i+1,r_j}} \{e_{r_1 \dots, \check{r}_p, \dots r_d}\}, \\ A_a^{\pm 1} * \{e_{r_1 \dots r_d}\} &= v^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \dots r_d}\} \quad \text{and} \\ B_a^{\pm 1} * \{e_{r_1 \dots r_d}\} &= v^{\mp \sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \dots r_d}\} \quad \text{and} \end{aligned}$$

Proof. The first two identities follow directly from Proposition 4.2.2. The last two identities are straightforward. \square

4.4. $\mathcal{H}_{\mathcal{Y}}$ -action on \mathcal{V} .

Definition 4.4.1. The two parameter Iwahori-Hecke algebra $\mathbf{H}_d(v, t)$ of type \mathbf{A}_d is a unital associative algebra over $\mathbb{Q}(v, t)$ generated by T_i for $i \in [1, d-1]$ and subject to the following relations.

$$\begin{aligned} T_i^2 &= (vt - v^{-1}t)T_i + t^2, \quad 1 \leq i \leq d-1, \\ T_j T_{j+1} T_j &= T_{j+1} T_j T_{j+1}, \quad 1 \leq j \leq d-2, \\ T_i T_j &= T_j T_i, \quad |i - j| > 1. \end{aligned}$$

We shall provide an explicit description of the action of $\mathcal{H}_{\mathcal{Y}}$ on \mathcal{V} . For any $1 \leq j \leq d-1$, we define a function T_j in $\mathcal{H}_{\mathcal{Y}}$ by

$$T_j(F, F') = \begin{cases} v^{-1}t, & \text{if } F_i = F'_i \ \forall i \in [1, d] \setminus \{j\}, F_j \neq F'_j; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.4.2. *The assignment of sending the functions T_j , for $1 \leq j \leq d-1$, in the algebra $\mathcal{H}_{\mathcal{Y}}$ to the generators of \mathbf{H}_d in the same notations is an isomorphism.*

Given $B = (b_{ij}) \in \Pi$, let r_c be the unique number in $[1, n]$ such that $b_{r_c, c} = 1$ for each $c \in [1, d]$. The correspondence $B \mapsto \tilde{\mathbf{r}} = (r_1, \dots, r_d)$ defines a bijection between Π and the set of all sequences (r_1, \dots, r_d) . Denote by $e_{r_1 \dots r_d}$ the characteristic function of the G -orbit corresponding to the matrix B in \mathcal{V} . It is clear that the collection of these characteristic functions provides a basis for \mathcal{V} .

Lemma 4.4.3. *The action of $\mathcal{H}_{\mathcal{Y}}$ on \mathcal{V} is described as follows. For $1 \leq j \leq d-1$, we have*

$$(18) \quad \{e_{r_1 \dots r_d}\} T_j = \begin{cases} \{e_{r_1 \dots r_{j-1} r_{j+1} r_j \dots r_d}\}, & r_j < r_{j+1}; \\ vt \{e_{r_1 \dots r_d}\}, & r_j = r_{j+1}; \\ (vt - v^{-1}t) \{e_{r_1 \dots r_d}\} + t^2 \{e_{r_1 \dots r_{j-1} r_{j+1} r_j \dots r_d}\}, & r_j > r_{j+1}. \end{cases}$$

Proof. Formula (18) similar with the one in [GL92, 1. 12], whose proof is also almost the same as one parameter of type- \mathbf{A} case. \square

4.5. Generators of \mathcal{S} . Define a partial order “ \leq ” on Θ_d by $\mathbf{a} \leq \mathbf{b}$ if $\mathcal{O}_{\mathbf{a}} \subset \overline{\mathcal{O}}_{\mathbf{b}}$. For any $\mathbf{a} = (a_{ij})$ and $\mathbf{b} = (b_{ij})$ in $\Xi_{\mathbf{d}}$, we say that $\mathbf{a} \preceq \mathbf{b}$ if and only if the following two conditions hold.

$$(19) \quad \sum_{r \leq i, s \geq j} a_{rs} \leq \sum_{r \leq i, s \geq j} b_{rs}, \quad \forall i < j.$$

$$(20) \quad \sum_{r \geq i, s \leq j} a_{rs} \leq \sum_{r \geq i, s \leq j} b_{rs}, \quad \forall i > j.$$

The relation “ \preceq ” defines a second partial order on Θ_d . We say that $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a} \preceq \mathbf{b}$ and at least one of the inequalities in (19) is strict. We shall denote by “ $\{\mathbf{m}\} + \text{lower terms}$ ” an element in \mathcal{S} which is equal to $\{\mathbf{m}\}$ plus a linear combination of $\{\mathbf{m}'\}$ with $\mathbf{m}' \prec \mathbf{m}$. By Proposition (4.2.2), we have

Corollary 4.5.1. *Assume that $1 \leq h < n$, $1 \leq h \leq n$, $M = (m_{i,j}) \in \theta_d$.*

(a) *Assume that $m_{h,j} = 0, \forall j > k, m_{h+1,j} = 0, \forall j \geq k$. Let $r = m_{h,k}$, $\mathbf{a} = (a_{ij}) \in \Xi_{\mathbf{d}}$ satisfies the following two conditions: $a_{h,k} = 0, a_{h+1,k} = r, a_{i,j} = m_{i,j}$ for all other i, j . If \mathbf{b} is subject to $\mathbf{b} - rE_{h,h+1}$ is diagonal, $\text{co}(\mathbf{b}) = \text{ro}(\mathbf{a})$, then*

$$\{\mathbf{b}\} * \{\mathbf{a}\} = \{M\} + \text{lower terms}.$$

(b) *Assume that $m_{h,j} = 0, \forall j \leq k, m_{h+1,j} = 0, \forall j < k$. Let $r = m_{h+1,k}$, $\mathbf{a} = (a_{ij}) \in \theta_d$ satisfies the following two conditions: $a_{h,k} = r, a_{h+1,k} = 0, a_{i,j} = m_{i,j}$ for all other i, j . If \mathbf{c} is subject to $\mathbf{c} - rE_{h,h+1}$ is diagonal, $\text{co}(\mathbf{c}) = \text{ro}(\mathbf{a})$, then*

$$\{\mathbf{c}\} * \{\mathbf{a}\} = \{M\} + \text{lower terms}.$$

Proof. In case (a), from the proof of the [BLM90, 3. 8], we have that $\{M\}$ is correspondence to $\mathbf{t} = (0, \dots, 0, R, 0, \dots, 0)$, where R is in the k place. Therefore, $\alpha(t) = \sum_{j \geq k} a_{h,j} t_k + \sum_{j > k} a_{h+1,k} t_k - \sum_{j < l} t_j t_l = 0$. Then (a) follows.

In case (b), we have we have that $\{M\}$ is correspondence to $\mathbf{t} = (0, \dots, 0, R, 0, \dots, 0)$, where R is in the k place. Therefore, $\alpha'(t) = \sum_{j \leq l} a_{h+1,j} t_l + \sum_{j < l} a_{h,j} t_l - \sum_{j < l} t_j t_l = 0$. Then (b) follows. \square

Theorem 4.5.2. *For any $\mathbf{a} = (a_{ij}) \in \Theta_{\mathbf{d}}$. The following identity holds in \mathcal{S}*

$$\prod_{1 \leq i \leq h < j \leq n} \{D_{i,h,j} + a_{i,j} E_{h,h+1}\} * \prod_{1 \leq j \leq h < i \leq n} \{D_{i,h,j} + a_{i,j} E_{h+1,h}\} = \{\mathbf{a}\} + \text{lower terms},$$

where the product is taken in the following order. The factors in the first product are taken in the following order: (i, h, j) comes before (i', h', j') if either $j > j'$ or $j = j', h - i < h' - i'$, or $j = j', h - i = h' - i', i' > i$. The factors in the second product are taken in the following order: (i, h, j) comes before (i', h', j') if either $i < i'$ or $i = i', h - j > h' - j'$, or $i = i', h - j = h' - j', j' < j$. The matrices $D_{i,h,j}$ are diagonal with entries in \mathbb{N} . Which are uniquely determined.

Proof. The proof of this theorem is similar to the [BLM90, 3. 9]. \square

We have immediately

Corollary 4.5.3. *The products $\mathbf{m}_{\mathbf{a}} = \prod_{1 \leq i \leq h < j \leq n} \{D_{i,h,j} + a_{i,j} E_{h,h+1}\} * \prod_{1 \leq j \leq h < i \leq n} \{D_{i,h,j} + a_{i,j} E_{h+1,h}\}$ for any $\mathbf{a} \in \Theta_{\mathbf{d}}$ in Theorem 4.5.2 form a basis for \mathcal{S} .*

By (16), (17) and Corollary 4.5.3, we have

Corollary 4.5.4. *The algebra \mathcal{S} (resp. $\mathbb{Q}(v) \otimes_{\mathcal{A}} \mathcal{S}$) is generated by the elements $[\mathfrak{e}]$ such that $\mathfrak{e} - RE_{i,i+1}$ (resp. either \mathfrak{e} or $\mathfrak{e} - RE_{i,i+1}$) is diagonal for some $R \in \mathbb{N}$ and $i \in [1, n-1]$.*

Observe that $E_i = \sum t\{\mathfrak{b}\}$, $F_i = \sum \{\mathfrak{c}\}$, $A_a^{\pm 1} = \sum v^{\pm d_a} t^{\pm d_a} \{\mathfrak{d}\}$, $B_a^{\pm 1} = \sum v^{\mp d_a} t^{\pm d_a} \{\mathfrak{d}\}$, $\forall i \in [1, n-1]$, $a \in [1, n]$, where \mathfrak{b} , \mathfrak{c} and \mathfrak{d} run over all matrices in $\Theta_{\mathbf{d}}$ such that $\mathfrak{b} - E_{i,i+1}$, $\mathfrak{c} - E_{i+1,i}$ and \mathfrak{d} are diagonal, respectively, and d_a is the (a, a) -entry of the matrix in \mathfrak{d} . We have the following corollary by Corollary 4.5.4.

Corollary 4.5.5. *The algebra $\mathbb{Q}(v, t) \otimes_{\mathcal{A}} \mathcal{S}$ is generated by the functions E_i , F_i , $A_a^{\pm 1}$, $B_a^{\pm 1}$ for any $i \in [1, n-1]$, $a \in [1, n]$.*

5. THE LIMIT ALGEBRA \mathcal{K}

5.1. Stabilization. Let I be the identity matrix. We set ${}_pA = A + pI$. Let $\tilde{\Theta}$ be the set of all $n \times n$ matrices with integer entries such that the entries off diagonal are ≥ 0 .

Let

$$\mathcal{K} = \text{span}_{\mathcal{A}} \{ \{\mathfrak{a}\} | \mathfrak{a} \in \tilde{\Theta} \},$$

where the notation $\{\mathfrak{a}\}$ is a formal symbol. Let v', t' be independent indeterminates, and we denote by \mathfrak{R} the ring $\mathbb{Q}(v, t)[v', t']$.

Proposition 5.1.1. *Suppose that $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_r$ ($r \geq 2$) are matrices in $\tilde{\Theta}$ such that $\text{co}(\mathfrak{a}_i) = \text{ro}(\mathfrak{a}_{i+1})$ for $1 \leq i \leq r-1$. There exist $\mathfrak{z}_1, \dots, \mathfrak{z}_m \in \tilde{\Theta}$, $G_j(v, v', t, t') \in \mathfrak{R}$ and $p_0 \in \mathbb{N}$ such that in \mathcal{S}_d for some d , we have*

$$[{}_p\mathfrak{a}_1] * [{}_p\mathfrak{a}_2] * \dots * [{}_p\mathfrak{a}_r] = \sum_{j=1}^m G_j(v, v^{-p}, t, t^p) [{}_p\mathfrak{z}_j], \quad \forall p \geq p_0.$$

Proof. The proof is essentially the same as the one for Proposition 4. 2 in [BLM90] by using Corollary 4.2.2 and Theorem 4.5.2. The main difference is that we should give how the twists $\alpha(t)$ and $\alpha'(t)$ change when \mathfrak{a} is replaced by ${}_p\mathfrak{a}$.

If $r = 2$ and \mathfrak{a}_1 is chosen such that $\mathfrak{a}_1 - RE_{h,h+1}$ is a diagonal with $R \in \mathbb{N}$, the structure constant $G_t(v, v', t, t')$ is defined by

$$G_t(v, v', t, t') = v^{\beta(t)} \prod_{\substack{1 \leq u \leq n \\ u \neq n}} \left(\frac{a_{h,u} + t_u}{t_u} \right)_v \prod_{1 \leq i \leq t_h} \frac{v^{-2(a_{h,h} + t_h - i)} v'^2 - 1}{v^{-2i} - 1} t^{\alpha(t)} t'^2 \sum_{h \geq u} t_u.$$

Similarly, if $r = 2$ and \mathfrak{a}_1 is chosen such that $\mathfrak{a}_1 - RE_{h+1,h}$ is diagonal with $R \in \mathbb{N}$, the structure constant $G_t(v, v', t, t')$ is defined by

$$G_t(v, v', t, t') = v^{\beta'(t)} \prod_{1 \leq u \leq n, u \neq h+1} \left(\frac{a_{h+1,u} + t_u}{t_u} \right)_v \prod_{1 \leq t \leq t_{h+1}} \frac{v^{-2(a_{h+1,h+1} + t_{h+1} - i + 1)} v'^2}{v^{-2i} - 1} t^{\alpha(t)} t' \sum_{h < u} t_u,$$

Keep in mind the above modifications, the rest of the proof for Proposition 4. 2 in [BLM90] can be repeated here. \square

By specialization v', t' at $v' = 1, t' = 1$, there is a unique associative \mathcal{A} -algebra structure on \mathcal{K} , without unit, where the product is given by

$$\{\mathbf{a}_1\} \cdot \{\mathbf{a}_2\} \cdots \{\mathbf{a}_r\} = \sum_{j=1}^m G_j(v, 1, t, 1)[\mathfrak{z}_j]$$

if $\mathbf{a}_1, \dots, \mathbf{a}_r$ are as in Proposition 5.1.1.

Let \mathbf{a} and $\mathbf{b} \in \tilde{\Theta}$ be chosen such that $\mathbf{b} - rE_{h,h+1}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{b}) = \text{ro}(\mathbf{a})$. Then we have

$$(21) \quad \{\mathbf{b}\} \cdot \{\mathbf{a}\} = \sum_t v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^N \overline{\binom{a_{hu} + t_u}{t_u}}_v \{\mathbf{a}_t\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^N$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{h+1,u}$ for all $u \neq h+1$, $\alpha(t), \beta(t), \mathbf{a}_t \in \tilde{\Theta}$ are defined in (16).

Similarly, if $\mathbf{a}, \mathbf{c} \in \tilde{\Theta}$ are chosen such that $\mathbf{c} - rE_{h+1,h}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{c}) = \text{ro}(\mathbf{a})$, then we have

$$(22) \quad \{\mathbf{c}\} \cdot \{\mathbf{a}\} = \sum_t v^{\beta'(t)} t^{\alpha'(t)} \prod_{u=1}^N \overline{\binom{a_{h+1,u} + t_u}{t_u}}_v \{\mathbf{a}(h, t)\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^N$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{h,u}$ for all $u \neq h$, $\alpha'(t), \beta'(t), \mathbf{a}(h, t) \in \tilde{\Theta}$ are defined in (17).

5.2. The algebra \mathcal{U} . In this section, we shall define a new algebra \mathcal{U} in the completion of \mathcal{K} similar to [BLM90, Section 5].

Let $\hat{\mathcal{K}}$ be the $\mathbb{Q}(v, t)$ -vector space of all formal sum $\sum_{\mathbf{a} \in \tilde{\Theta}} \xi_{\mathbf{a}} \{\mathbf{a}\}$ with $\xi_{\mathbf{a}} \in \mathbb{Q}(v, t)$ and a locally finite property, i. e., for any $\mathbf{t} \in \mathbb{Z}^n$, the sets $\{\mathbf{a} \in \tilde{\Theta} | \text{ro}(\mathbf{a}) = \mathbf{t}, \xi_{\mathbf{a}} \neq 0\}$ and $\{\mathbf{a} \in \tilde{\Theta} | \text{co}(\mathbf{a}) = \mathbf{t}, \xi_{\mathbf{a}} \neq 0\}$ are finite. The space $\hat{\mathcal{K}}$ becomes an associative algebra over $\mathbb{Q}(v, t)$ when equipped with the following multiplication:

$$\sum_{\mathbf{a} \in \tilde{\Xi}_{\mathbf{D}}} \xi_{\mathbf{a}} \{\mathbf{a}\} \cdot \sum_{\mathbf{b} \in \tilde{\Xi}_{\mathbf{D}}} \xi_{\mathbf{b}} \{\mathbf{b}\} = \sum_{\mathbf{a}, \mathbf{b}} \xi_{\mathbf{a}} \xi_{\mathbf{b}} \{\mathbf{a}\} \cdot \{\mathbf{b}\},$$

where the product $\{\mathbf{a}\} \cdot \{\mathbf{b}\}$ is taken in \mathcal{K} .

Observe that the algebra $\hat{\mathcal{K}}$ has a unit element $\sum \{\mathfrak{d}\}$, the summation of all diagonal matrices.

We define the following elements in $\hat{\mathcal{K}}$. For any nonzero matrix $\mathbf{a} \in \tilde{\Theta}$, let $\hat{\mathbf{a}}$ be the matrix obtained by replacing diagonal entries of \mathbf{a} by zeroes. We set

$$\Theta^0 = \{\hat{\mathbf{a}} | \mathbf{a} \in \tilde{\Theta}\}.$$

For any $\hat{\mathbf{a}}$ in Θ^0 and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we define

$$(23) \quad \hat{\mathbf{a}}(\mathbf{j}) = \sum_{\lambda} v^{\lambda_1 j_1 + \dots + \lambda_n j_n} t^{\lambda_1 |j_1| + \dots + \lambda_n |j_n|} \{\hat{\mathbf{a}} + D_{\lambda}\}$$

where the sum runs through all $\lambda = (\lambda_i) \in \mathbb{Z}^n$ such that $\hat{\mathbf{a}} + D_{\lambda} \in \tilde{\Theta}$, where D_{λ} is the diagonal matrices with diagonal entries (λ_i) .

For $i \in [1, n-1]$, let

$$E_i = E_{i,i+1}(0) \quad \text{and} \quad F_i = E_{i+1,i}(0).$$

Let \mathcal{U} be the subalgebra of $\hat{\mathcal{K}}$ generated by $E_i, F_i, 0(\mathbf{j})$ for all $i \in [1, n-1]$ and $\mathbf{j} \in \mathbb{Z}^n$.

Proposition 5.2.1. *The following relations hold in \mathcal{U} .*

$$\begin{aligned} (24) \quad & 0(\mathbf{j})0(\mathbf{j}') = 0(\mathbf{j}')0(\mathbf{j}), \\ (25) \quad & 0(\mathbf{j})E_h = v^{j_h-j_{h+1}}t^{|j_h|-|j_{h+1}|}E_h0(\mathbf{j}), \quad 0(\mathbf{j})F_h = v^{-j_h+j_{h+1}}t^{-|j_h|+|j_{h+1}|}F_h0(\mathbf{j}), \\ (26) \quad & t(E_hF_h - F_hE_h) = (v - v^{-1})^{-1}(0(\underline{h} - \underline{h+1}) - 0(\underline{h+1} - \underline{h})), \\ (27) \quad & E_i^2E_{i+1} - (vt + v^{-1}t)E_iE_{i+1}E_i + t^2E_{i+1}E_i^2 = 0, \\ (28) \quad & t^2E_{i+1}^2E_i - (vt + v^{-1}t)E_{i+1}E_iE_{i+1} + E_iE_{i+1}^2 = 0, \\ (29) \quad & F_i^2F_{i+1} - (vt^{-1} + v^{-1}t^{-1})F_iF_{i+1}F_i + t^{-2}F_{i+1}F_i^2 = 0, \\ (30) \quad & t^{-2}F_{i+1}^2F_i - (vt^{-1} + v^{-1}t^{-1})F_{i+1}F_iF_{i+1} + F_iF_{i+1}^2 = 0. \end{aligned}$$

where $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}^n$, $h, i, j \in [1, n]$ and $\underline{i} \in \mathbb{N}^N$ is the vector whose i -th entry is 1 and 0 elsewhere.

Proof. We show (25).

$$\begin{aligned} 0(\mathbf{j})E_h &= \sum_{\lambda} v^{\sum \lambda_k j_k} t^{\sum \lambda_k |j_k|} \{D_{\lambda}\} \sum_{\lambda'} \{E_{h,h+1} + D_{\lambda'}\} \\ &= \sum_{\lambda'} v^{\sum \lambda'_k j_k + j_h} t^{\sum \lambda'_k |j_k| + |j_h|} \{E_{h,h+1} + D_{\lambda'}\}, \end{aligned}$$

where the sums run through in an obvious range by the definition in (23).

$$\begin{aligned} E_h0(\mathbf{j}) &= \sum_{\lambda, \lambda'} v^{\sum \lambda_k j_k} t^{\sum \lambda_k |j_k|} \{E_{h,h+1} + D_{\lambda'}\} \{D_{\lambda}\} \\ &= \sum_{\lambda'} v^{\sum \lambda'_k j_k + j_{h+1}} t^{\sum \lambda'_k |j_k| + |j_{h+1}|} \{E_{h,h+1} + D_{\lambda'}\}. \end{aligned}$$

So we have the first identity in (25). All other identities in (24) and (25) can be shown similarly.

We show (26). we have

$$\begin{aligned} E_hF_h &= \sum_{\lambda, \lambda'} \{E_{h,h+1} + D_{\lambda}\} \{E_{h+1,h} + D_{\lambda'}\} \\ &= \sum_{\lambda} \{E_{h,h+1} + D_{\lambda}\} \{E_{h+1,h} + D_{\lambda}\} \\ &= \sum_{\lambda} (v^{\lambda_h - \lambda_{h+1}} t^{\lambda_h + \lambda_{h+1}} \overline{\binom{\lambda_h + 1}{1}}_v \{D_{\lambda} + E_{h,h}\} \\ &\quad + \{E_{h+1,h} + E_{h,h+1} + D_{\lambda} - E_{h+1,h+1}\}). \end{aligned}$$

Similarly,

$$\begin{aligned} F_hE_h &= \sum_{\lambda} \{E_{h+1,h} + D_{\lambda}\} \{E_{h,h+1} + D_{\lambda}\} \\ &= \sum_{\lambda} (v^{\lambda_{h+1} - \lambda_h} t^{\lambda_h + \lambda_{h+1}} \overline{\binom{\lambda_{h+1} + 1}{1}}_v \{D_{\lambda} + E_{h+1,h+1}\} \\ &\quad + \{E_{h+1,h} + E_{h,h+1} + D_{\lambda} - E_{h,h}\}). \end{aligned}$$

Therefore,

$$\begin{aligned} t(E_hF_h - F_hE_h) &= \sum_{\lambda} \frac{v^{\lambda_h - \lambda_{h+1}} t^{\lambda_h + \lambda_{h+1}} - v^{\lambda_{h+1} - \lambda_h} t^{\lambda_h + \lambda_{h+1}}}{v - v^{-1}} \{D_{\lambda}\} \\ &= (v - v^{-1})^{-1}(0(\underline{h} - \underline{h+1}) - 0(\underline{h+1} - \underline{h})). \end{aligned}$$

At last, We show (27).

$$\begin{aligned} E_h^2 E_{h+1} &= \sum_{\lambda} vt(v^{-2} + 1) \{D_{\lambda} + E_{h,h+1} + E_{h,h+2}\} \\ &\quad + \sum_{\lambda} v^{-1} t^3 (v^{-2} + 1) \{D_{\lambda} + E_{h+1,h+2} + 2E_{h,h+1}\}; \end{aligned}$$

$$\begin{aligned} E_h E_{h+1} E_h &= \sum_{\lambda} t^2 (v^{-2} + 1) \{D_{\lambda} + 2E_{h,h+1} + E_{h,h+2}\} \\ &\quad + \sum_{\lambda} \{D_{\lambda} + E_{h,h+1} + E_{h,h+2}\}; \end{aligned}$$

$$E_{h+1} E_h^2 = \sum_{\lambda} vt(v^{-2} + 1) \{D_{\lambda} + 2E_{h,h+1} + E_{h+1,h+2}\}.$$

Then the first identity of 27 follows. all other identities can be shown similarly. \square

The Corollary directly follows.

Corollary 5.2.2. *The assignment $E_i \mapsto tE_i$, $F_i \mapsto F_i$, $A_a \mapsto 0(\underline{a})$ and $B_a \mapsto 0(-\underline{a})$, for any $i \in [1, n-1]$, $a \in [1, n]$, defines a algebra isomorphism $\Upsilon : U_{v,t}(gl_n) \rightarrow \mathcal{U}$.*

6. SCHUR DUALITIES FOR TWO PARAMETER CASE OF TYPE \mathbf{A}_d

In this section, we shall formulate algebraically the dualities between algebras $U_{v,t}(gl_n)$ and the two parameter Iwahori-Hecke algebras $H_d(v, t)$ of type \mathbf{A}_d .

Let \mathbf{V} be a vector space over $\mathbb{Q}(v, t)$ of dimension n . We fix a basis $(\mathbf{v}_i)_{1 \leq i \leq n}$ for \mathbf{V} . Let $\mathbf{V}^{\otimes d}$ be the d -th tensor space of \mathbf{V} . Thus we have a basis $(\mathbf{v}_{r_1} \otimes \cdots \otimes \mathbf{v}_{r_d})$, where $r_1, \dots, r_d \in [1, n]$, for the tensor space $\mathbf{V}^{\otimes d}$.

For a sequence $\mathbf{r} = (r_1, \dots, r_d)$, we write $\mathbf{v}_{\mathbf{r}}$ for $\mathbf{v}_{r_1} \otimes \cdots \otimes \mathbf{v}_{r_d}$.

For a sequence \mathbf{r} and a fixed integer $p \in [1, d]$, we define the sequence \mathbf{r}'_p and \mathbf{r}''_p by

$$(\mathbf{r}'_p)_j = \begin{cases} r_j, & j \neq p, \\ r_p - 1, & j = p \end{cases} \quad \text{and} \quad (\mathbf{r}''_p)_j = \begin{cases} r_j, & j \neq p, \\ r_p + 1, & j = p, \end{cases}$$

Lemma 6.0.3. *There has a left $U_{v,t}(gl_n)$ -action on $\mathbf{V}^{\otimes d}$ defined by, for any $i \in [1, n-1]$, $a \in [1, n]$,*

$$\begin{aligned} E_i \cdot \mathbf{v}_{\mathbf{r}} &= \sum_{1 \leq p \leq d: r_p = i+1} v^{\sum_{j>p} \delta_{i,r_j} - \delta_{i+1,r_j}} t^{1 + \sum_{j>p} \delta_{i,r_j} + \delta_{i+1,r_j}} \mathbf{v}_{\mathbf{r}'_p}, \\ F_i \cdot \mathbf{v}_{\mathbf{r}} &= \sum_{1 \leq p \leq d: r_p = i} v^{\sum_{j<p} \delta_{i+1,r_j} - \delta_{i,r_j}} t^{\sum_{j<p} \delta_{i,r_j} + \delta_{i+1,r_j}} \mathbf{v}_{\mathbf{r}''_p}, \\ A_a^{\pm 1} \cdot \mathbf{v}_{\mathbf{r}} &= v^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} \mathbf{v}_{\mathbf{r}}, \\ B_a^{\pm 1} \cdot \mathbf{v}_{\mathbf{r}} &= v^{\mp \sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\pm \sum_{1 \leq j \leq d} \delta_{a,r_j}} \mathbf{v}_{\mathbf{r}}. \end{aligned}$$

The lemma follows Proposition 4.1.1, and Corollary 4.3.1.

Lemma 6.0.4. *There has a right \mathbf{H}_d -action on $\mathbf{V}^{\otimes d}$ given by, for $1 \leq j \leq d-1$,*

$$(31) \quad \mathbf{v}_{r_1 \dots r_d} T_j = \begin{cases} \mathbf{v}_{r_1 \dots r_{j-1} r_{j+1} r_j \dots r_d}, & r_j < r_{j+1}; \\ vt \mathbf{v}_{r_1 \dots r_d}, & r_j = r_{j+1}; \\ (vt - v^{-1}t) \mathbf{v}_{r_1 \dots r_d} + t^2 \mathbf{v}_{r_1 \dots r_{j-1} r_{j+1} r_j \dots r_d}, & r_j > r_{j+1}. \end{cases}$$

This lemma follows Lemmas 4.4.2 and 4.4.3.

We now can state the duality.

Proposition 6.0.5. *The left $U_{v,t}(gl_n)$ -action in Lemma 6.0.3 and the right \mathbf{H}_d -action in Lemma 4.4.3 on $\mathbf{V}^{\otimes d}$ are commuting. They form a double centralizer for $n \geq d$, i. e. ,*

$$\mathbf{H}_d \simeq \text{End}_{\mathbf{U}}(\mathbf{V}^{\otimes d}) \quad \text{and} \quad U_{v,t}(gl_n) \rightarrow \text{End}_{\mathbf{H}_d}(\mathbf{V}^{\otimes d}) \text{ is surjective.}$$

The proposition follows from the previous two lemmas, Lemma 3.1.1, Proposition 4.1.1 and Corollary 4.5.5.

6.1. Galois descend approach. Let $G = \text{Gal}(\mathbb{Q}(v, t)/\mathbb{Q}(r, s))$, $r = vt, s = v^{-1}t$. It is easy to know $G \cong S_2$ which is generated by σ . G act on $U_{v,t}(gl_n)$ given by a \mathbb{Q} algebra homomorphism $\sigma : U_{v,t}(gl_n) \rightarrow U_{v,t}(gl_n)$; $E_i \mapsto -E_i, F_i \mapsto F_i, K_i \mapsto K_i, K'_i \mapsto K'_i, v \mapsto -v, t \mapsto -t$. G can be also act on $V^{\otimes k}$ which is given by $\sigma : V^{\otimes k} \rightarrow V^{\otimes k}$; $v_{i_1} \otimes \dots \otimes v_{i_k} \mapsto v_{i_1} \otimes \dots \otimes v_{i_k}, v \mapsto -v, t \mapsto -t$. By the directly compute. we have the following lemma.

Lemma 6.1.1. *The G -actions on $(U_{v,t}(gl_n), V^{\otimes k})$ is compatible. That is $\sigma(av) = \sigma(a)\sigma(v)$, $\forall a \in U_{v,t}(gl_n), v \in V$.*

Proof. We only need to check the identities $\sigma(av) = \sigma(a)\sigma(v)$ on the generators. By the lemma 6.0.3. The result is obvious. \square

Though the above lemma we know there is a G -action on $H_k(v, t)$ which is given by $\sigma : H_k(v, t) \mapsto H_k(v, t); T_i \mapsto T_i, v \mapsto -v, t \mapsto -t$.

Theorem 6.1.2. *$(U_{v,t}(gl_n)^G, V^{\otimes k G}, H_k(v, t)^G)$ is a shur-weyl tripple. and $U_{v,t}(gl_n)^G \cong U_{r,s}(gl_n)$, $V^{\otimes k G}$ is a n^k dimension vector space over $\mathbb{Q}(r, s)$, $H_k(v, t)^G \cong H_k(r, s)$.*

Proof. \square

Remark 6.1.3. $H_k(r, s)$ is a unital associate algebra over $\mathbb{Q}(r, s)$ with generators \widetilde{T}_i , $1 \leq i < k$ subject to the following relations:

- (1) $\widetilde{T}_i \widetilde{T}_{i+1} \widetilde{T}_i = \widetilde{T}_{i+1} \widetilde{T}_i \widetilde{T}_{i+1}, 1 \leq i < k.$
- (2) $\widetilde{T}_i \widetilde{T}_j = \widetilde{T}_j \widetilde{T}_i, \text{ if } |i - j| \geq 2.$
- (3) $(\widetilde{T}_i - r)(\widetilde{T}_i + s) = 0, \forall i.$

$U_{r,s}(gl_n)$ is a $\mathbb{Q}(r, s)$ algebra generated by $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i, \widetilde{K}'_i$.

7. TWO NEW QUANTUM GROUP $\widetilde{U_{v,t}(gl_n)^m}$ AND $\widehat{U_{v,t}(gl_n)^m}$

In order to give the comultiplication in the two parameter case of two new quantum group appeared in [FL14], we give two new quantum group $\widetilde{U_{v,t}(gl_n)^m}$ and $\widehat{U_{v,t}(gl_n)^m}$ in this section.

For any $i \in [1, n-1]$, $a \in [1, n]$, $m \in [1, n-1]$, we define the function $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$ to be the same function in \mathcal{S} . we further define

$$(32) \quad \begin{aligned} J_+(V, V') &= \begin{cases} 1, & \text{if } V = V' \text{ and } |V_m| = d \bmod 2; \\ 0, & \text{otherwise.} \end{cases} \\ J_-(V, V') &= \begin{cases} 1, & \text{if } V = V' \text{ and } |V_m| = d-1 \bmod 2; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

All these functions are elements in \mathcal{S} .

Proposition 7.0.4. *The functions $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$ and J_{\pm} in \mathcal{S} , for any $i \in [1, n-1]$, $a \in [1, n]$, satisfy the relations in 4.1.1 together with the following relations.*

$$(R1) \quad \begin{aligned} J_+ + J_- &= 1, \quad J_{\alpha} J_{\beta} = \delta_{\alpha, \beta} J_{\alpha}, \quad J_{\pm} A_a = A_a J_{\pm}, \quad J_{\pm} A_a = A_a J_{\pm}, \\ J_{\pm} E_i &= E_i J_{\pm}, \quad J_{\pm} F_i = F_i J_{\pm}, i \neq m; \quad J_{\pm} E_m = E_m J_{\mp}, J_{\pm} F_m = F_m J_{\mp}; \end{aligned}$$

Corollary 7.0.5. *The algebra $\mathbb{Q}(v, t) \otimes_{\mathcal{A}} \mathcal{S}$ is generated by the functions $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$, and J_{\pm} in \mathcal{S} , for any $i \in [1, n-1]$, $a \in [1, n]$.*

7.1. Another limit algebra \mathcal{K}' . We set ${}_p A = A + 2pI$. Let

$$\mathcal{K}' = \text{span}_{\mathcal{A}} \{ \{ \mathbf{a} \} | \mathbf{a} \in \tilde{\Theta} \},$$

where the notation $\{ \mathbf{a} \}$ is a formal symbol. Let v', t' be a independent indeterminates, and we denote by \mathfrak{R} the ring $\mathbb{Q}(v, t)[v', t']$.

Proposition 7.1.1. *Suppose that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$ ($r \geq 2$) are matrices in $\tilde{\Theta}$ such that $\text{co}(\mathbf{a}_i) = \text{ro}(\mathbf{a}_{i+1})$ for $1 \leq i \leq r-1$. There exist $\mathfrak{z}_1, \dots, \mathfrak{z}_m \in \tilde{\Theta}$, $G'_j(v, v', t, t') \in \mathfrak{R}$ and $p_0 \in \mathbb{N}$ such that in \mathcal{S}_d for some d , we have*

$$\{ {}_p \mathbf{a}_1 \} * \{ {}_p \mathbf{a}_2 \} * \dots * \{ {}_p \mathbf{a}_r \} = \sum_{j=1}^m G'_j(v, v^{-p}, t, t^p) \{ {}_p \mathfrak{z}_j \}, \quad \forall p \geq p_0.$$

By specialization v', t' at $v' = 1, t' = 1$, there is a unique associative \mathcal{A} -algebra structure on \mathcal{K} , without unit, where the product is given by

$$\{ \mathbf{a}_1 \} \cdot \{ \mathbf{a}_2 \} \cdot \dots \cdot \{ \mathbf{a}_r \} = \sum_{j=1}^m G'_j(v, 1, t, 1) [\mathfrak{z}_j]$$

if $\mathbf{a}_1, \dots, \mathbf{a}_r$ are as in Proposition 7.1.1.

Let \mathbf{a} and $\mathbf{b} \in \tilde{\Theta}$ be chosen such that $\mathbf{b} - rE_{m, m+1}$ is diagonal for some $r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{b}) = \text{ro}(\mathbf{a})$. Then we have

$$(33) \quad \{ \mathbf{b} \} \cdot \{ \mathbf{a} \} = \sum_t v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^N \overline{\binom{a_{hu} + t_u}{t_u}}_v \{ \mathbf{a}_t \},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^n$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{m+1,u}$, $\alpha(t), \beta(t), \mathbf{a}_t \in \tilde{\Theta}$ are defined in (16), .

Similarly, if $\mathbf{a}, \mathbf{c} \in \tilde{\Theta}$ are chosen such that $\mathbf{c} - rE_{m+2,m+1}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{c}) = \text{ro}(\mathbf{a})$, then we have

$$(34) \quad \{\mathbf{c}\} \cdot \{\mathbf{a}\} = \sum_t v^{\beta'(t)} t^{\alpha'(t)} \prod_{u=1}^N \overline{\binom{a_{h+1,u} + t_u}{t_u}}_v \{\mathbf{a}(h, t)\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^n$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{m+1,u}$, $\alpha'(t), \beta'(t)$ $\mathbf{a}(h, t) \in \tilde{\Theta}$ are defined in (17).

7.2. The algebra \mathcal{U}' . In this section, we shall define a new algebra \mathcal{U} in the completion of \mathcal{K} similar to [BLM90, Section 5].

Let $\hat{\mathcal{K}}$ be the $\mathbb{Q}(v, t)$ -vector space of all formal sum $\sum_{\mathbf{a} \in \tilde{\Theta}} \xi_{\mathbf{a}} \{\mathbf{a}\}$ with $\xi_{\mathbf{a}} \in \mathbb{Q}(v, t)$ and a locally finite property, i. e. , for any $\mathbf{t} \in \mathbb{Z}^n$, the sets $\{\mathbf{a} \in \tilde{\Theta} | \text{ro}(\mathbf{a}) = \mathbf{t}, \xi_{\mathbf{a}} \neq 0\}$ and $\{\mathbf{a} \in \tilde{\Theta} | \text{co}(\mathbf{a}) = \mathbf{t}, \xi_{\mathbf{a}} \neq 0\}$ are finite. The space $\hat{\mathcal{K}}$ becomes an associative algebra over $\mathbb{Q}(v, t)$ when equipped with the following multiplication:

$$\sum_{\mathbf{a} \in \tilde{\Theta}} \xi_{\mathbf{a}} \{\mathbf{a}\} \cdot \sum_{\mathbf{b} \in \tilde{\Theta}} \xi_{\mathbf{b}} \{\mathbf{b}\} = \sum_{\mathbf{a}, \mathbf{b}} \xi_{\mathbf{a}} \xi_{\mathbf{b}} \{\mathbf{a}\} \cdot \{\mathbf{b}\},$$

where the product $\{\mathbf{a}\} \cdot \{\mathbf{b}\}$ is taken in \mathcal{K} . This is shown in exactly the same as [BLM90, Section 5].

Observe that the algebra $\hat{\mathcal{K}}$ has a unit element $\sum \{\mathbf{d}\}$, the summation of all diagonal matrices.

We define the following elements in $\hat{\mathcal{K}}$. For any nonzero matrix $\mathbf{a} \in \tilde{\Theta}$, let $\hat{\mathbf{a}}$ be the matrix obtained by replacing diagonal entries of \mathbf{a} by zeroes. We set $\Theta^0 = \{\hat{\mathbf{a}} | \mathbf{a} \in \tilde{\Theta}\}$.

For any $\hat{\mathbf{a}}$ in Θ^0 and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we define

$$(35) \quad \hat{\mathbf{a}}(\mathbf{j}) = \sum_{\lambda} v^{\lambda_1 j_1 + \dots + \lambda_n j_n} t^{\lambda_1 |j_1| + \dots + \lambda_n |j_n|} \{\hat{\mathbf{a}} + D_{\lambda}\}$$

where the sum runs through all $\lambda = (\lambda_i) \in \mathbb{Z}^n$ such that $\hat{\mathbf{a}} + D_{\lambda} \in \tilde{\Theta}$, where D_{λ} is the diagonal matrices with diagonal entries (λ_i) .

And we also define

$$J_+ = \sum_{\lambda \in S_0} \{D_{\lambda}\}, \quad J_- = \sum_{\lambda \in S_1} \{D_{\lambda}\},$$

Where $S_0 = \{\lambda | \sum_{i=1}^m \lambda_i \equiv \sum_{i=1}^n \lambda_i \pmod{2}\}$, $S_1 = \{\lambda | \sum_{i=1}^m \lambda_i \equiv \sum_{i=1}^n \lambda_i - 1 \pmod{2}\}$

For $i \in [1, n-1]$, let

$$E_i = E_{i,i+1}(0) \quad \text{and} \quad F_i = E_{i+1,i}(0).$$

Let \mathcal{U}' be the subalgebra of $\hat{\mathcal{K}}$ generated by $E_i, F_i, 0(\mathbf{j})$, J_{\pm} for all $i \in [1, n-1]$ and $\mathbf{j} \in \mathbb{Z}^n$.

Proposition 7.2.1. *The following relations hold in \mathcal{U}' .*

$$(36) \quad J_+ + J_- = 1, \quad J_{\alpha} J_{\beta} = \delta_{\alpha, \beta} J_{\alpha}, \quad J_{\pm} 0(\mathbf{j}) = 0(\mathbf{j}) J_{\pm},$$

$$(37) \quad J_{\pm} E_i = E_i J_{\pm}, \quad J_{\pm} F_i = F_i J_{\pm}, i \neq m; \quad J_{\pm} E_m = E_m J_{\mp}, J_{\pm} F_m = F_m J_{\mp};$$

$$(38) \quad 0(\mathbf{j}) 0(\mathbf{j}') = 0(\mathbf{j}') 0(\mathbf{j}),$$

$$(39) \quad 0(\mathbf{j})E_h = v^{j_h-j_{h+1}}t^{|j_h|-|j_{h+1}|}E_h0(\mathbf{j}), \quad 0(\mathbf{j})F_h = v^{-j_h+j_{h+1}}t^{-|j_h|+|j_{h+1}|}F_h0(\mathbf{j}),$$

$$(40) \quad t(E_hF_h - F_hE_h) = (v - v^{-1})^{-1}(0(\underline{h} - \underline{h} + 1) - 0(\underline{h} + 1 - \underline{h})),$$

$$(41) \quad E_i^2E_{i+1} - (vt + v^{-1}t)E_iE_{i+1}E_i + t^2E_{i+1}E_i^2 = 0,$$

$$(42) \quad t^2E_{i+1}^2E_i - (vt + v^{-1}t)E_{i+1}E_iE_{i+1} + E_iE_{i+1}^2 = 0,$$

$$(43) \quad F_i^2F_{i+1} - (vt^{-1} + v^{-1}t^{-1})F_iF_{i+1}F_i + t^{-2}F_{i+1}F_i^2 = 0,$$

$$(44) \quad t^{-2}F_{i+1}^2F_i - (vt^{-1} + v^{-1}t^{-1})F_{i+1}F_iF_{i+1} + F_iF_{i+1}^2 = 0.$$

where $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}^n$, $h, i, j \in [1, n]$ and $\underline{i} \in \mathbb{N}^N$ is the vector whose i -th entry is 1 and 0 elsewhere.

7.3. The algebra $\widetilde{U_{v,t}(gl_n)^m}$.

Definition 7.3.1. $\widetilde{U_{v,t}(gl_n)^m}$ is an associative $\mathbb{Q}(v, t)$ -algebra with 1 generated by symbols $E_i, F_i, A_a, B_a, J_\alpha$ for all $i \in [1, n-1]$, $a \in [1, n]$ and $\alpha \in \{+, -\}$ and subject to the following relations.

$$(45) \quad J_+ + J_- = 1, \quad J_\alpha J_\beta = \delta_{\alpha,\beta} J_\alpha, \quad J_\pm A_a = A_a J_\pm, \quad J_\pm B_a = B_a J_\pm,$$

$$(46) \quad J_\pm E_i = E_i J_\pm, \quad J_\pm F_i = F_i J_\pm, i \neq m; \quad J_\pm E_m = E_m J_\mp, J_\pm F_m = F_m J_\mp;$$

$$(47) \quad A_i^{\pm 1} A_j^{\pm 1} = A_j^{\pm 1} A_i^{\pm 1}, \quad B_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} B_i^{\pm 1},$$

$$(48) \quad A_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} A_i^{\pm 1}, \quad A_i^{\pm 1} A_i^{\mp 1} = 1 = B_i^{\pm 1} B_i^{\mp 1}.$$

$$(49) \quad A_i E_j A_i^{-1} = v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \quad B_i E_j B_i^{-1} = v^{-\langle i, j \rangle} t^{\langle i, j \rangle} E_j,$$

$$(50) \quad A_i F_j A_i^{-1} = v^{-\langle i, j \rangle} t^{-\langle j, i \rangle} F_j, \quad B_i F_j B_i^{-1} = v^{\langle i, j \rangle} t^{-\langle j, i \rangle} F_j.$$

$$(51) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{vt - v^{-1}t}.$$

$$(52) \quad E_i^2 E_{i+1} - (vt + v^{-1}t) E_i E_{i+1} E_i + t^2 E_{i+1} E_i^2 = 0,$$

$$(53) \quad t^2 E_{i+1}^2 E_i - (vt + v^{-1}t) E_{i+1} E_i E_{i+1} + E_i E_{i+1}^2 = 0,$$

$$(54) \quad F_i^2 F_{i+1} - (vt^{-1} + v^{-1}t^{-1}) F_i F_{i+1} F_i + t^{-2} F_{i+1} F_i^2 = 0,$$

$$(55) \quad t^{-2} F_{i+1}^2 F_i - (vt^{-1} + v^{-1}t^{-1}) F_{i+1} F_i F_{i+1} + F_i F_{i+1}^2 = 0.$$

Proposition 7.3.2. The assignment $E_i \mapsto E_i$, $F_i \mapsto F_i$, $A_a \mapsto 0(\underline{a})$, $B_a \mapsto 0(-\underline{a})$, and $J_\alpha \mapsto J_\alpha$ for any $i \in [1, n-1]$, $a \in [1, n]$, $\alpha \in \{+, -\}$ defines a algebra isomorphism $\Upsilon : \widetilde{U_{v,t}(gl_n)^m} \rightarrow \mathcal{U}'$.

7.4. Defining relations of \mathcal{S} . For any $i \in [1, n-1]$, $a \in [1, n]$, $m \in [1, n-1]$, we define the function $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$ to be the same function in \mathcal{S} . we further define

$$(56) \quad J_+(V, V') = \begin{cases} 1, & \text{if } V_m = V_{m+1}, |V_m| \equiv d \pmod{2}; \\ 0, & \text{otherwise.} \end{cases}$$

$$J_-(V, V') = \begin{cases} 1, & \text{if } V_m = V_{m+1}, |V_m| \equiv d-1 \pmod{2}; \\ 0, & \text{otherwise.} \end{cases}$$

$$J_0 = 1 - J_+ - J_-.$$

Proposition 7.4.1. *The functions $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$, and J_α in \mathcal{S} , for any $i \in [1, n-1]$, $a \in [1, n]$, $\alpha \in \{+, -, 0\}$, satisfy the relations in proposition 4.1.1 and the following relations.*

$$(57) \quad J_+ + J_0 + J_- = 1, J_\alpha J_\beta = \delta_{\alpha, \beta} J_\alpha, J_\alpha A_a = A_a J_\alpha, J_\alpha B_a = A_a B_\alpha;$$

$$(58) \quad E_i J_\pm = (1 - \delta_{i, m}) J_\pm E_i, J_\pm E_i = (1 - \delta_{i, m+1}) E_i J_\pm;$$

$$(59) \quad F_i J_\pm = (1 - \delta_{i, m+1}) J_\pm F_i, J_\pm F_i = (1 - \delta_{i, m}) F_i J_\pm;$$

$$(60) \quad J_\pm E_m E_{m+1} = E_m E_{m+1} J_\mp;$$

$$(61) \quad J_\pm F_{m+1} F_m = F_{m+1} F_m J_\mp;$$

$$(62) \quad J_\pm E_m F_m - E_m F_m J_\mp = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_\pm - J_\mp);$$

$$(63) \quad J_\pm F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_\mp = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_\pm - J_\mp).$$

Proof. The first identity in the first three rows of the relations in the proposition are straightforward. Let $\lambda'_i = |V'_i/V'_{i-1}|$. We show the identity 61, by a direct calculation. We have

$$F_{m+1} F_m(V, V') = \begin{cases} v^{-\lambda'_{m+2} - \lambda'_{m+1}} t^{\lambda'_{m+2} + \lambda'_{m+1}}, & \text{if } V_m \subseteq V'_m \text{ and } V_{m+1} \subseteq V'_{m+1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$J_+ F_{m+1} F_m(V, V') = \begin{cases} v^{-\lambda'_{m+2} - \lambda'_{m+1}} t^{\lambda'_{m+2} + \lambda'_{m+1}}, & \text{if } V_m \subseteq V'_m, V_{m+1} \subseteq V'_{m+1}, V_m = V_{m+1} \\ & \text{and } |V_m| \equiv d \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$F_{m+1} F_m J_-(V, V') = \begin{cases} v^{-\lambda'_{m+2} - \lambda'_{m+1}} t^{\lambda'_{m+2} + \lambda'_{m+1}}, & \text{if } V_m \subseteq V'_m, V_{m+1} \subseteq V'_{m+1}, V_m = V_{m+1} \\ & \text{and } |V_m| \equiv d \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$$

The first part of the identity 61 follows, all other identities in 61 and 60 can be shown similarly.

Then, We show the identity 63. By a direct calculation, we have

$$F_{m+1} E_{m+1}(V, V') = \begin{cases} \frac{v^{2\lambda'_{m+2}-1}}{v^2-1} v^{-\lambda'_{m+2} - \lambda'_{m+1} + 1} t^{\lambda'_{m+2} + \lambda'_{m+1}}, & \text{if } V = V' \\ v^{-\lambda'_{m+2} - \lambda'_{m+1} + 1} t^{\lambda'_{m+2} + \lambda'_{m+1}}, & \text{if } |V_{m+1} \cap V'_{m+1}| = |V_{m+1}| - 1 = |V'_{m+1}| - 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$(J_+ F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_-)(V, V') = \begin{cases} \frac{v^{\lambda'_{m+2}} t^{\lambda'_{m+2}} - v^{-\lambda'_{m+2}} t^{\lambda'_{m+2}}}{v - v^{-1}}, & \text{if } V = V', V_m = V_{m+1}, \\ & \text{and } |V_m| \equiv d \pmod{2}, \\ \frac{v^{-\lambda'_{m+2}} t^{\lambda'_{m+2}} - v^{\lambda'_{m+2}} t^{\lambda'_{m+2}}}{v - v^{-1}}, & \text{if } V = V', V_m = V_{m+1}, \\ & \text{and } |V_m| \equiv d - 1 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{B_{m+1}A_{m+2} - A_{m+1}B_{m+2}}{v - v^{-1}}(J_{\pm} - J_{\mp})(V, V') = \begin{cases} \frac{v^{\lambda'_{m+2}}t^{\lambda'_{m+2}} - v^{-\lambda'_{m+2}}t^{\lambda'_{m+2}}}{v - v^{-1}}, & \text{if } V = V', V_m = V_{m+1}, \\ & \text{and } |V_m| \equiv d \pmod{2}, \\ \frac{v^{-\lambda'_{m+2}}t^{\lambda'_{m+2}} - v^{\lambda'_{m+2}}t^{\lambda'_{m+2}}}{v - v^{-1}}, & \text{if } V = V', V_m = V_{m+1}, \\ & \text{and } |V_m| \equiv d - 1 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$$

The first part of the identity 63 follows, all other identities in 63 and 62 can be shown similarly. \square

Corollary 7.4.2. *The algebra $\mathbb{Q}(v, t) \otimes_{\mathcal{A}} \mathcal{S}$ is generated by the functions $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}$, and J_{α} in \mathcal{S} , for any $i \in [1, n-1]$, $a \in [1, n]$, $\alpha \in \{+, -, 0\}$.*

7.5. Limit algebra \mathcal{K}'' . Let $I' = I - E_{m+1, m+1}$ be the identity matrix. We set ${}_p A = A + 2pI'$. Let $\tilde{\Theta}' = \{M | M \in \tilde{\Theta}, M_{m+1, m+1} \geq 0\}$.

Let

$$\mathcal{K}'' = \text{span}_{\mathcal{A}}\{\{\mathbf{a}\} | \mathbf{a} \in \tilde{\Theta}'\},$$

where the notation $\{\mathbf{a}\}$ is a formal symbol bearing no geometric meaning. Let v', t' be a independent indeterminates, and we denote by \mathfrak{R} the ring $Q(v, t)[v', t']$.

Proposition 7.5.1. *Suppose that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$ ($r \geq 2$) are matrices in $\tilde{\Theta}'$ such that $\text{co}(\mathbf{a}_i) = \text{ro}(\mathbf{a}_{i+1})$ for $1 \leq i \leq r-1$. There exist $\mathfrak{z}_1, \dots, \mathfrak{z}_m \in \tilde{\Theta}'$, $G'_j(v, v', t, t') \in \mathfrak{R}$ and $p_0 \in \mathbb{N}$ such that in \mathcal{S}_d for some d , we have*

$$\{{}_p \mathbf{a}_1\} * \{{}_p \mathbf{a}_2\} * \dots * \{{}_p \mathbf{a}_r\} = \sum_{j=1}^m G'_j(v, v^{-p}, t, t^p) \{{}_p \mathfrak{z}_j\}, \quad \forall p \geq p_0.$$

By specialization v', t' at $v' = 1, t' = 1$, there is a unique associative \mathcal{A} -algebra structure on \mathcal{K} , without unit, where the product is given by

$$\{\mathbf{a}_1\} \cdot \{\mathbf{a}_2\} \cdot \dots \cdot \{\mathbf{a}_r\} = \sum_{j=1}^m G'_j(v, 1, t, 1) [\mathfrak{z}_j]$$

if $\mathbf{a}_1, \dots, \mathbf{a}_r$ are as in Proposition 7.5.1.

Let \mathbf{a} and $\mathbf{b} \in \tilde{\Theta}$ be chosen such that $\mathbf{b} - rE_{m, m+1}$ is diagonal for some $r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{b}) = \text{ro}(\mathbf{a})$. Then we have

$$(64) \quad \{\mathbf{b}\} \cdot \{\mathbf{a}\} = \sum_t v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^N \overline{\binom{a_{hu} + t_u}{t_u}}_v \{\mathbf{a}_t\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^n$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{m+1, u}$, $\alpha(t), \beta(t)$, $\mathbf{a}_t \in \tilde{\Theta}'$ are defined in (16).

Similarly, if $\mathbf{a}, \mathbf{c} \in \tilde{\Theta}$ are chosen such that $\mathbf{c} - rE_{m+2, m+1}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $\text{co}(\mathbf{c}) = \text{ro}(\mathbf{a})$, then we have

$$(65) \quad \{\mathbf{c}\} \cdot \{\mathbf{a}\} = \sum_t v^{\beta'(t)} t^{\alpha'(t)} \prod_{u=1}^N \overline{\binom{a_{h+1, u} + t_u}{t_u}}_v \{\mathbf{a}(h, t)\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^n$ such that $\sum_{u=1}^n t_u = r$ and $t_u \leq a_{m+1,u}$, $\alpha'(t), \beta'(t)$ $\mathfrak{a}(h, t) \in \tilde{\Theta}'$ are defined in (17).

7.6. The algebra \mathcal{U}'' . In this section, we shall define a new algebra \mathcal{U}'' in the completion of \mathcal{K}'' similar to [BLM90, Section 5].

Let $\hat{\mathcal{K}}''$ be the $\mathbb{Q}(v, t)$ -vector space of all formal sum $\sum_{\mathfrak{a} \in \tilde{\Theta}'} \xi_{\mathfrak{a}} \{\mathfrak{a}\}$ with $\xi_{\mathfrak{a}} \in \mathbb{Q}(v, t)$ and a locally finite property, i. e. , for any $\mathbf{t} \in \mathbb{Z}^n$, the sets $\{\mathfrak{a} \in \tilde{\Theta}' | \text{ro}(\mathfrak{a}) = \mathbf{t}, \xi_{\mathfrak{a}} \neq 0\}$ and $\{\mathfrak{a} \in \tilde{\Theta}' | \text{co}(\mathfrak{a}) = \mathbf{t}, \xi_{\mathfrak{a}} \neq 0\}$ are finite. The space $\hat{\mathcal{K}}''$ becomes an associative algebra over $\mathbb{Q}(v, t)$ when equipped with the following multiplication:

$$\sum_{\mathfrak{a} \in \tilde{\Theta}'} \xi_{\mathfrak{a}} \{\mathfrak{a}\} \cdot \sum_{\mathfrak{b} \in \tilde{\Theta}'} \xi_{\mathfrak{b}} \{\mathfrak{b}\} = \sum_{\mathfrak{a}, \mathfrak{b}} \xi_{\mathfrak{a}} \xi_{\mathfrak{b}} \{\mathfrak{a}\} \cdot \{\mathfrak{b}\},$$

where the product $\{\mathfrak{a}\} \cdot \{\mathfrak{b}\}$ is taken in \mathcal{K}'' . This is shown in exactly the same as [BLM90, Section 5].

Observe that the algebra $\hat{\mathcal{K}}''$ has a unit element $\sum \{\mathfrak{d}\}$, the summation of all diagonal matrices.

We define the following elements in $\hat{\mathcal{K}}''$. For any nonzero matrix $\mathfrak{a} \in \tilde{\Theta}'$, let $\hat{\mathfrak{a}}$ be the matrix obtained by replacing diagonal entries of \mathfrak{a} by zeroes. We set

$$\Theta^0 = \{\hat{\mathfrak{a}} | \mathfrak{a} \in \tilde{\Theta}'\}.$$

For any $\hat{\mathfrak{a}}$ in Θ^0 and $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n$, we define

$$(66) \quad \hat{\mathfrak{a}}(\mathbf{j}) = \sum_{\lambda} v^{\lambda_1 j_1 + \dots + \lambda_n j_n} t^{\lambda_1 |j_1| + \dots + \lambda_n |j_n|} \{\hat{\mathfrak{a}} + D_{\lambda}\}$$

where the sum runs through all $\lambda = (\lambda_i) \in \mathbb{Z}^n$ such that $\hat{\mathfrak{a}} + D_{\lambda} \in \tilde{\Theta}'$, where D_{λ} is the diagonal matrices with diagonal entries (λ_i) .

And we also define

$$(67) \quad J_+ = \sum_{\lambda \in S_0} \{D_{\lambda}\},$$

$$(68) \quad J_- = \sum_{\lambda \in S_1} \{D_{\lambda}\},$$

$$(69) \quad J_0 = 1 - J_+ - J_-.$$

Where $S_0 = \{\lambda | \lambda_{m+1} = 0, \sum_{i=1}^m \lambda_i \equiv \sum_{i=1}^n \lambda_i \pmod{2}\}$, $S_1 = \{\lambda | \lambda_{m+1} = 0, \sum_{i=1}^m \lambda_i \equiv \sum_{i=1}^n \lambda_i - 1 \pmod{2}\}$

For $i \in [1, n-1]$, let

$$E_i = E_{i, i+1}(0) \quad \text{and} \quad F_i = E_{i+1, i}(0).$$

Let \mathcal{U} be the subalgebra of $\hat{\mathcal{K}}$ generated by $E_i, F_i, 0(\mathbf{j}), J_{\alpha}$ for all $i \in [1, n-1]$, $\mathbf{j} \in \mathbb{Z}^n$ and $\alpha \in \{+, -, 0\}$.

Proposition 7.6.1. *The following relations hold in \mathcal{U}'' .*

$$(70) \quad J_+ + J_0 + J_- = 1, J_{\alpha} J_{\beta} = \delta_{\alpha, \beta} J_{\alpha}, J_{\alpha} A_a = A_a J_{\alpha}, J_{\alpha} B_a = B_a J_{\alpha};$$

$$(71) \quad E_i J_{\pm} = (1 - \delta_{i, m}) J_{\pm} E_i, J_{\pm} E_i = (1 - \delta_{i, m+1}) E_i J_{\pm};$$

$$(72) \quad F_i J_{\pm} = (1 - \delta_{i, m+1}) J_{\pm} F_i, J_{\pm} F_i = (1 - \delta_{i, m}) F_i J_{\pm};$$

$$(73) \quad J_{\pm} E_m E_{m+1} = E_m E_{m+1} J_{\mp};$$

$$(74) \quad J_{\pm} F_{m+1} F_m = F_{m+1} F_m J_{\mp};$$

$$(75) \quad J_{\pm} E_m F_m - E_m F_m J_{\mp} = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_{\pm} - J_{\mp});$$

$$(76) \quad J_{\pm} F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_{\mp} = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_{\pm} - J_{\mp}).$$

$$(77) \quad 0(\mathbf{j})0(\mathbf{j}') = 0(\mathbf{j}')0(\mathbf{j}),$$

$$(78) \quad 0(\mathbf{j})E_h = v^{j_h - j_{h+1}} t^{|j_h| - |j_{h+1}|} E_h 0(\mathbf{j}), \quad 0(\mathbf{j})F_h = v^{-j_h + j_{h+1}} t^{-|j_h| + |j_{h+1}|} F_h 0(\mathbf{j}),$$

$$(79) \quad t(E_h F_h - F_h E_h) = (v - v^{-1})^{-1} (0(\underline{h} - \underline{h} + 1) - 0(\underline{h} + 1 - \underline{h})), h \neq m + 1$$

$$(80) \quad E_i^2 E_{i+1} - (vt + v^{-1}t) E_i E_{i+1} E_i + t^2 E_{i+1} E_i^2 = 0,$$

$$(81) \quad t^2 E_{i+1}^2 E_i - (vt + v^{-1}t) E_{i+1} E_i E_{i+1} + E_i E_{i+1}^2 = 0,$$

$$(82) \quad F_i^2 F_{i+1} - (vt^{-1} + v^{-1}t^{-1}) F_i F_{i+1} F_i + t^{-2} F_{i+1} F_i^2 = 0,$$

$$(83) \quad t^{-2} F_{i+1}^2 F_i - (vt^{-1} + v^{-1}t^{-1}) F_{i+1} F_i F_{i+1} + F_i F_{i+1}^2 = 0.$$

where $\mathbf{j}, \mathbf{j}' \in \mathbb{Z}^n$, $h, i, j \in [1, n]$ and $\underline{i} \in \mathbb{N}^n$ is the vector whose i -th entry is 1 and 0 elsewhere.

7.7. The algebra $\widehat{U_{v,t}(gl_N)^m}$.

Definition 7.7.1. $\widehat{U_{v,t}(gl_n)}$ is an associative $\mathbb{Q}(v, t)$ -algebra with 1 generated by symbols $E_i, F_i, J_{\alpha}, A_a, B_a$ for all $i \in [1, n-1]$, $a \in [1, n]$ and $\alpha \in \{+, -, 0\}$ and subject to the following relations.

$$(84) \quad J_+ + J_0 + J_- = 1, J_{\alpha} J_{\beta} = \delta_{\alpha, \beta} J_{\alpha}, J_{\alpha} A_a = A_a J_{\alpha}, J_{\alpha} B_a = B_a J_{\alpha};$$

$$(85) \quad E_i J_{\pm} = (1 - \delta_{i,m}) J_{\pm} E_i, J_{\pm} E_i = (1 - \delta_{i,m+1}) E_i J_{\pm};$$

$$(86) \quad F_i J_{\pm} = (1 - \delta_{i,m+1}) J_{\pm} F_i, J_{\pm} F_i = (1 - \delta_{i,m}) F_i J_{\pm};$$

$$(87) \quad J_{\pm} E_m E_{m+1} = E_m E_{m+1} J_{\mp};$$

$$(88) \quad J_{\pm} F_{m+1} F_m = F_{m+1} F_m J_{\mp};$$

$$(89) \quad J_{\pm} E_m F_m - E_m F_m J_{\mp} = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_{\pm} - J_{\mp});$$

$$(90) \quad J_{\pm} F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_{\mp} = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_{\pm} - J_{\mp}).$$

$$(91) \quad A_i^{\pm 1} A_j^{\pm 1} = A_j^{\pm 1} A_i^{\pm 1}, \quad B_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} B_i^{\pm 1},$$

$$(92) \quad A_i^{\pm 1} B_j^{\pm 1} = B_j^{\pm 1} A_i^{\pm 1}, \quad A_i^{\pm 1} A_i^{\mp 1} = 1 = B_i^{\pm 1} B_i^{\mp 1}.$$

$$(93) \quad A_i E_j A_i^{-1} = v^{\langle i, j \rangle} t^{\langle i, j \rangle} E_j, \quad B_i E_j B_i^{-1} = v^{-\langle i, j \rangle} t^{\langle i, j \rangle} E_j,$$

$$(94) \quad A_i F_j A_i^{-1} = v^{-\langle i, j \rangle} t^{-\langle i, j \rangle} F_j, \quad B_i F_j B_i^{-1} = v^{\langle i, j \rangle} t^{-\langle i, j \rangle} F_j.$$

$$(95) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{v - v^{-1}}.$$

$$(96) \quad E_i^2 E_{i+1} - (vt + v^{-1}t) E_i E_{i+1} E_i + t^2 E_{i+1} E_i^2 = 0,$$

$$(97) \quad t^2 E_{i+1}^2 E_i - (vt + v^{-1}t) E_{i+1} E_i E_{i+1} + E_i E_{i+1}^2 = 0,$$

$$(98) \quad F_i^2 F_{i+1} - (vt^{-1} + v^{-1}t^{-1}) F_i F_{i+1} F_i + t^{-2} F_{i+1} F_i^2 = 0,$$

$$(99) \quad t^{-2} F_{i+1}^2 F_i - (vt^{-1} + v^{-1}t^{-1}) F_{i+1} F_i F_{i+1} + F_i F_{i+1}^2 = 0.$$

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1. DEPARTMENT OF MATHEMATICS, SOUTH CHINA UNIVERSITY OF TECHNOLOGY, GUANGZHOU, CHINA 510641

E-mail address: Zhengzj@scut.edu.cn

2. DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MAHATTAN, KANSAS 66506

E-mail address: zlin@math.ksu.edu